THEORETICAL MECHANICS

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FIRST LESSONS

IN

THEORETICAL MECHANICS

BY THE

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PREFACE.

In writing the following pages, I have had in view the wants of a rather numerous class of readers—those who wish to study the first principles of mechanics before they have obtained the knowledge of geometry, algebra, and trigonometry, which most elementary books on the subject presuppose. Some knowledge of arithmetic and geometry, it is true, must be assumed in discussing the most elementary questions as to forces; but I have found on trial that a very large portion of the principles of mechanics admits of exposition and illustration without demanding of the student a knowledge of more than arithmetic, a few rules in mensuration, enough geometry to make accurate diagrams with compasses, scale, and protractor, and enough algebra to solve a simple equation. No more than this is needed for the study of the following pages, with the exception of Chap. VI., *on motion in a circle*, and a few articles and examples, occurring, for the most part, towards the end of the book.

It will be proper to observe here that a good deal of choice has been exercised both as to the contents of the book, and as to the order in which they have been arranged.

Thus, the methods of finding centres of gravity and moments of inertia being in reality branches of geometry, have not been treated more fully than was necessary for the purposes of the present volume. Again, any fact or theorem has been taken for granted without proof whenever this seemed to conduce to the clear exposition of the matter in hand, e.g. the principle of the parallelogram of forces has been assumed without proof in Chap. III., though further on in the volume Newton's proof is given. However, most of the theorems of elementary mechanics are demonstrated, as this was found to be the most convenient way of putting them effectually before the student. In discussing the equilibrium of a body, the cases in which it is acted on by two or three forces are fully considered, but very little more has been attempted, partly on account of the great importance of the second case—that of three forces—which marks it out as calling for a full discussion; partly because a complete treatment of the case of n forces would have unduly lengthened, and have been inconsistent with the purpose of the present work. The answers to most of the examples and questions of the third chapter are intended to be obtained by diagrams drawn to scale; even if the student knows enough trigonometry to calculate them, he will find it a most useful exercise to obtain them graphically—a method which admits of extensive development, and is, I believe, largely employed by engineers. In the chapter on motion in a circle, something more is given than the motion of a point; the treatment is necessarily imperfect, but I hope enough is done to throw some light on the motion round a fixed axis of a body symmetrical to a plane at right angles to the axis—the case that commonly occurs in machinery.

It is not necessary to specify here the contents of the book; I will only add that I have endeavoured to explain as clearly as possible the leading ideas of the subject, to illustrate them by a great number of examples and questions, and to get rid of all difficulties that are not inherent in the subject. How far I have succeeded is another question; but in case any of my readers find the book hard to understand, it may be well to add that when all that is possible has been done in the way of exposition and illustration, the subject will still present difficulties to most beginners; in fact, it does not admit of an easy and familiar treatment, and 'is not to be understood without that degree of attention which the very nature of the thing requires.'

J. F. TWISDEN.

July, 1874.

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Elementary mechanics, like elementary geometry, is a study accessible to all: but, like that too, or perhaps more than that, it is a study which requires effort and contention of mind—a forced steadiness of thought. It is long since one complained of this labour in geometry, and was answered that in that region there is no Royal Road. The same is true of Mechanics, and must be true of all branches of solid education. But we should express the truth more appropriately in our days by saying that there is no Popular Road to these sciences. In the mind, as in the body, strenuous exercise alone can give strength and activity. The art of exact thought can be acquired only by the labour of close thinking.

W. WHEWELL.

THEORETICAL MECHANICS.

CHAPTER I.

INTRODUCTORY.

1. *Matter* is generally defined as that which affects our senses, as that on which force can be exerted, or that which can exert force: thus *air*, though invisible, affects our senses in various ways; it can be set in motion and form a current of air; it can exert force, e.g. it can turn the vanes of a windmill; consequently air is a matter. The like may be said of water, wood, stone, iron, &c. Now, however unlike each other air, water, and iron may be, they have several properties in common, and when regarded under the view of these common properties, it is convenient—perhaps necessary—to have a common name for them all. That common name is matter.

A limited portion of matter is called a *body*, e.g. a lump of lead is a body. When a body is so small that for the purposes of any discussion the relative positions of its parts need not be considered, it is spoken of as a *material point*, or a *heavy point*, or simply as a *point*.

2. *Quantity of matter*.—When two bodies, placed one in each pan of a perfectly just balance, exactly counterpoise each other, they contain equal quantities of

matter; and the two together contain twice the quantity of matter that is contained in either. It appears, therefore, that quantity of matter is ascertained by weighing. There is in the Exchequer Office in London a lump of platinum which is the standard pound, and there are four authorised copies of it kept in other places,[1] so that the standard could be replaced if lost, injured, or destroyed. This standard pound is the unit by which the quantity of matter in any given body is estimated. To say that a body weighs 10 lbs., is to say that the body placed in a perfectly just balance would exactly counterpoise ten bodies each of which would separately counterpoise the standard pound. As a matter of convenience other denominations are used in estimating quantity of matter; thus:—112 lbs. are called a hundredweight, 2240 lbs. a ton, the 1-16th part of a pound is called an ounce, and the 1-7000th part of a pound a grain. By a pound is meant a pound avoirdupois, unless the contrary is specified.

3. *Density.*—It is a point of common observation that equal volumes of different substances have unequal weights. It may be assumed, therefore, that the matter in the heavier bodies is more closely packed together than in the lighter bodies. Thus, 8 cubic inches of dry oak weigh about as much as 1 cubic inch of cast iron; it is assumed that in cast iron the matter is packed together about 8 times as closely as in dry oak, and cast iron is therefore said to have about 8 times the density of dry oak. In most cases it is found that portions of a given body having equal volumes have also equal weights; thus, if 2 cubic inches of steel are taken from any different parts of the same ingot, they will have nearly or exactly the same weight: when this is the case the body is said to

[1] Viz., the Mint, the Royal Society, Greenwich Observatory, and the House of Parliament. B. Stewart, on *Heat*, pp. 66, 69.

be of uniform density. It is to be assumed that a body is of a uniform density unless the contrary is expressly stated. By the density of a substance is meant the quantity of matter contained in the unit of volume of the substance.

4. *Density of water.*—It has been found as the result of very careful weighing that the density of distilled water at a given temperature is uniformly constant. At a temperature of 60° F., when both water and weights are in vacuó, a cubic inch of distilled water weighs 252·769[1] grains. It follows from this—and the student should verify the statements—that (*a*) a cubic foot of water weighs nearly 1000 oz. or 62·5 lbs., (*b*) a cubic yard of water is slightly less than $\frac{3}{4}$ths of a ton.

5. *Specific gravity.*—The specific gravity or specific density of a substance is the ratio which its density bears to the density of some standard substance. In the case of solids and liquids the standard substance is distilled water at 60° F.[2] To say that the specific gravity of cast iron is 7·2, is to say that any volume of cast iron weighs 7·2 times as much as an equal volume of water—strictly speaking, both the iron and the distilled water being at the standard temperature. The specific gravities of a large number of substances have been determined with great accuracy. The following table gives approximate values of the specific gravity of some substances; the values given in the table will be of use in working the examples that occur in the course of the present work.

TABLE I.

Platinum	21·5	Lead	11·4
Gold	19·25	Silver	10·5
Mercury	13·5	Copper	8·8

[1] B. Stewart, on *Heat*, p. 70.

[2] It is necessary to specify some temperature, as bodies expand or contract, i.e. their densities decrease or increase with change of temperature. Any temperature might be taken as a standard.

Brass	8·3	Mahogany	0·85
Wrought iron and steel	7·8	Teak	0·7
Cast iron	7·2	Beech	0·7
Glass	2·7	Elm	0·6
Granite	2·6	Fir	0·6
Sandstone	2·4	Cork	0·24
Brick	2·	Olive oil	0·92
Coal	1·3	Petroleum	0·88
Oak	0·9	Pure alcohol	0·79

This table enables us to determine approximately the weight of bodies of known size without weighing them. For this purpose the weight of a cubic foot of water may be assumed to be 1000 oz. Consequently the specific gravity of a substance multiplied by 1000 gives approximately the weight of a cubic foot of that substance in ounces, e.g. the weight of a cubic foot of steel is about 7800 oz., the weight of a cubic foot of granite is about 2600 oz. Hence we have the following rule:—From the known dimensions of the body find its volume in cubic feet; multiply the volume in cubic feet by 1000 times the specific gravity of the substance; the product is the weight of the body in ounces.

Ex. 1.—What is the weight of a plate of wrought iron 16 ft. long, 2 ft. wide, and 9 in. deep?

The volume is $16 \times 2 \times \frac{3}{4}$, or 24 cubic feet; the weight is therefore 24×7800 oz., or about 5 tons.

6. *Remark.*—The foregoing table of specific gravities gives only average and approximate values; consequently the specific gravity of a given specimen of any one substance, if determined by direct experiment, would probably not agree exactly with the registered value. In the case of a simple substance such as mercury, when it is in a state of purity and when precautions are taken against avoidable errors, a constant value for the specific gravity should be obtained; and it may be mentioned that when both water and mercury have a temperature of 62° F., the specific

gravity of mercury has been found to equal 13·569.[1] In cases where the substance is not simple constancy must not be looked for, e.g. cast iron contains foreign substances, such as carbon, sulphur, phosphorus, in a greater or less degree, and accordingly when the specific gravities of 16 specimens of cast iron were determined it was found that the highest value was 7·295 and the lowest 6·953.[2] In like manner the specific gravity of flint glass is said to be sometimes as high as 3·78 and that of plate glass as low as 2·37.[3] With different kinds of wood the variations are considerable and depend partly on growth, partly on seasoning; thus the specific gravity of Dantzig oak is said to be as low as 0·756,[4] while in the case of very old heart of oak it is said to be as high as 1·17.[5]

7. *Force.*—It is a fundamental fact that if a point were left wholly to itself it would either continue at rest or would move uniformly in a straight line. Any cause which changes or tends to change either of these states is a force. When a point at rest is observed to begin to move we infer that a force has acted on the point. When a point in motion is observed to move either more or less quickly, or to change the direction of its motion, in either case we infer that a force has acted on the point. If a piece of cork with a needle in it floats on water and a magnet is brought near it the cork moves towards the magnet; the magnet, therefore, exerts force on the body. If a bullet is held in the hand and is then let go, it falls towards the earth with a continually increasing velocity; the earth therefore exerts force on the body; and that force is called *gravity*.

There are other kinds of force; thus, the resistance

[1] B. Stewart, on *Heat*, p. 71.

[2] Moseley, *Mechanical Principles of Engineering*, p. 624.

[3] Müller's *Lehrbuch der Physik*, p. 14.

[4] Moseley, *Mechanical Principles of Engineering*, p. 624.

[5] Young, *Natural Philosophy*, vol. ii. p. 505.

offered by a solid body to being stretched, compressed, or twisted, is force; the resistance offered by a fluid to the motion of a body through it is force; friction is force; the muscular power of men or animals is force; the effort of a gas to expand is force. A force may be exerted without actually producing motion or change in direction of motion, because it is counteracted by another force or forces.

8. *Specification of a force.*—A force is specified when we know (*a*) the point on which it acts, (*b*) the line along which it acts, (*c*) the direction along the line, (*d*) its magnitude. The conditions (*a*) (*b*) and (*c*) present no difficulty. When a bullet is let fall, it moves along a vertical line downward; gravity therefore acts, (*a*) on the bullet, (*b*) along a vertical line, (*c*) and downward along that line. The term *direction* of a force is often used so as to comprise both the points (*b*) and (*c*). This use of the word is often convenient and need not—at least in many cases—cause ambiguity. The student must, however, bear in mind that it is one thing to say that a force acts along a certain line, and another to say that it acts from right to left along the line, and that it is often necessary to keep these points distinct. The fourth condition (*d*) presents more difficulty; but it may be overcome by observing that the attraction of the earth at a given place on a given quantity of matter is a constant force. We may therefore measure forces in units based on this fact as follows:—

A force of one pound is the attraction of the earth on a pound of matter in London at the sea level. Hence when we speak of a force of one pound, we mean a force that would by direct opposition support a pound of matter in London against gravity; a force of two pounds would just support two pounds of matter under like circumstances; and so on. The unit we have just defined is

called the *gravitation unit*, to distinguish it from the *absolute unit*, which will be defined further on.

9. *The force of gravity* is different at different places. It must be particularly noticed that the attraction of the earth on a given body is not exactly the same at different places. If the standard pound were weighed in a spring balance at a place near the equator, it would be found to be about 22 grains lighter than if similarly weighed in London. If it were possible to remove the balance to a height of about 4000 miles above the earth's surface, the compression of the spring would be reduced to about one-fourth part of what it was on the earth's surface, i.e. the attraction of the earth on a pound of matter in such a position is about the same as that exerted on a quarter of a pound of matter on the earth's surface. For this reason it is necessary in defining the gravitation unit to specify the place at which the determination is made; and though in a large number of questions the attraction of the earth on a pound of matter may be taken as a force of one pound without qualification, yet the student must bear in mind that a mass of one pound and a force of one pound are two totally distinct things. The former is simply a body which will just counterpoise a certain standard lump of matter in a perfectly just balance, the earth's attraction being of any amount whatever, provided it is equally exerted on both bodies. The latter is the actual amount of the attraction exerted on the body in some specified place. In short, the *mass* of the body is something that the body is in itself, and continues unchanged when the body is moved from place to place; the *weight* of the body is the force exerted on it by the earth and is different in different places.

10. *Action and reaction.*—When one body (A) acts on another body (B) the action is mutual; that is to say, B's reaction on A is a force equal and opposite to that with

which A acts on B. If I press my right hand with a force of 5 lbs. on my left hand, the left hand presses back against the right with an equal force of 5 lbs.; if a body weighing 20 lbs. is placed on a horizontal table, it presses downward on the table with a force of 20 lbs., while the table reacts upward against the body with a force of 20 lbs. If we suppose the surfaces of the bodies in contact to be perfectly smooth, the mutual action can be exerted only along the common perpendicular to the two surfaces at the point of contact; thus, let A B (fig. 1) be a smooth plane, and M a mass urged against it by any given forces; draw the line R M R′ at right angles to A B, the action (R′) of M on the plane can only be exerted along M R′, and the reaction

FIG. 1.

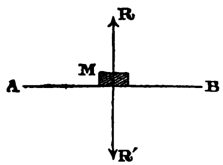
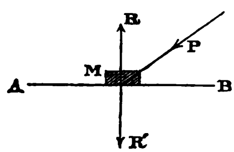


FIG. 2.

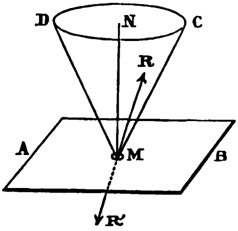


(R) of the plane against the body can only be exerted along M R, and these forces (R and R′) are equal. It will be observed that the forces acting on M are the given forces and R, e.g. if M is urged against the plane by a single force P, as shown in fig. 2, the forces really acting on M are P and the reaction (R) of A B; R balances the part of P which presses the body against the plane; the other part of P is employed in causing M to move.

11. *The angle of friction.*—If we suppose the surfaces of contact to be rough, instead of smooth as in the last article, the mutual action will take place along a line inclined to the common perpendicular at some angle or other not greater than a certain angle called the *angle of friction.* Thus, let A B be a rough plane, and M a mass

urged against it by certain forces; draw M N at right angles to A B, and make the angle N M C equal to the angle of friction; with M N as axis and N M C as semi-vertical angle, describe the cone C M D; the action of M on the plane and the reaction of the plane on M must take place along some line not falling without the cone, e.g. draw the line R M R′, then if the action takes place along M R′, the reaction will be exerted along M R. It deserves particular notice that the actual direction of M R will depend on the circumstances of the question—all that can be stated generally is that M R will not be outside the cone. If M is on the point of sliding, the line of the mutual action will be inclined to the perpendicular at an angle equal to the angle of friction, and the reaction of the plane will be opposed to the sliding of M.

FIG. 3.

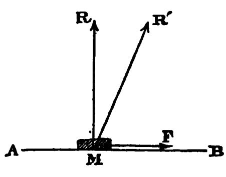


If we suppose the forces acting on bodies to be considerably less than what would crush them, the magnitude of the angle of friction is nearly constant for given materials. For example, suppose the surfaces planed but not made artificially smooth, and the material to be metal pressed against metal, the angle of friction is about 10°; thus, if M were copper and A B were iron, the angle N M C would be about 10°, and the mutual action could take place along any line M R, provided N M R were not greater than 10°. If the surfaces are wood on wood, the angle of friction is about 18°; if brick or stone on brick or stone, about 33°; if the contact is between metal and metal, wood and wood, or metal and wood, but an unguent, such as tallow, hog's lard, olive oil, &c., is interposed, the angle of friction is reduced to about 5°. The angle of friction is sometimes called the *angle of repose*, some-

times the *limiting angle of resistance*; but these names all mean the same thing.

12. *Coefficient of friction.*—In order to complete our view of the roughness of surfaces we will anticipate a point that will be explained further on:—Let M be a mass urged against the plane A B by certain forces, and let the reaction of the plane (R′) be exerted along M R′; the force R′ may be replaced by two forces R and F, one acting perpendicularly to and the other along A B. R, which is called the normal reaction, equals the force with which M is urged directly against the plane; F, which is called the friction, opposes the tendency of the body to slide in the direction B to A. If we suppose the forces acting on M to be such that the body is on the point of sliding, it is easily shown that F equals R × tangent of angle of friction; or, as it is more commonly written, $F = \mu R$; the multiplier μ, or the tangent of the angle of friction, is called the coefficient of friction.

FIG. 4.



If the surfaces in contact are metal on metal, the average value of the coefficient of friction is tan 10° or 0·18; if wood on wood, tan 18° or 0·33; if brick or stone on brick or stone, tan 33° or 0·65; if unguents are interposed, tan 5° or 0·1. If the surfaces are actually in motion, the coefficient of friction is in many cases less than before the motion begins. It is particularly to be observed that for a given normal reaction (R), the friction actually exerted will be the force necessary to prevent sliding, provided sliding can be prevented by a force less than μ R. It is only when sliding is about to take place or actually takes place that the whole amount μ R is called into play. Friction always act in a direction opposite to that in which the body slides or tends to slide. The

following propositions are called the laws of friction; they are approximately true when the forces do not exceed tolerably wide limits, and when the surfaces slide, or are on the point of sliding:—

(1) Friction is proportional to normal pressure.

(2) Friction is independent of the extent of the surfaces in contact.

(3) Friction is independent of the velocity of the motion.[1]

13. *Resultant and components.*—When several forces act on a body, one force can, in most cases, be found mechanically equivalent to them; that force is called their *resultant*, e.g. the weight of a body is due to the attraction of the earth on each of the small parts of which we may conceive the body to be made up; the weight of the whole body is consequently the resultant of the weights of all its parts. Similarly when a body is in contact with another body, e.g. when it rests on a table, it will probably touch the table at many points; at each point the table will react on the body, and consequently what is called the reaction of the table is in reality the resultant of the reactions at the several points.

When forces act on a point or body there are methods by which their resultant, if there be one, can be found;

[1] The numerical values of the angles of friction given in the text are taken from p. 398 of Prof. Willis's *Principles of Mechanism*; on the same page is given a graphic representation of M. Morin's experimental results, which shows the differences between actual values and the mean values given above. It will be observed that no estimate of the angle is given in the case of metal on wood; this is because the values in the case of various metals and woods are so different that an average would be misleading; e.g. wrought iron on oak has a coefficient of friction equal to 0·62, and an angle equal to 32°, while cast iron on elm has a coefficient and angle equal to 0·20 and 11°. The variations for different kinds of wood are very considerable. It appears, on comparing Mr. Willis's diagram with M. Morin's tables (*Notions fondamentales*, p. 307), that the diagram and the averages refer to the case in which motion actually occurs.

on the other hand, a single force can be resolved into a number of forces, which are jointly equivalent to it and are called the *components* of that force.

14. *Forms of matter.*—For our present purpose we may consider two kinds of bodies, *solids* and *fluids*. Solids may be either *rigid*, as an iron crowbar, or *flexible*, as a silk thread; fluids may be either *incompressible*, as water, or *compressible*, as air. We will consider the point with regard to these four examples :—

(*a*) When an iron crowbar is said to be rigid we mean that under the action of the forces ordinarily brought to bear upon it, its change of form is either insensibly small, or at all events so small as to render the consideration of it unnecessary. It is only in this sense that pieces of wood, stone, metal, &c. are spoken of as rigid bodies. The distinctive property of a rigid body is that it transmits a force in the direction of the force and in that direction only, whether it be a pull or a push; thus, if P acts on a point A, and a second point B is taken in its line of action, we may suppose P to act indifferently at A or B as may be most convenient, provided A and B are points of a rigid body, or, as it is sometimes stated, provided they are rigidly connected.

(*b*) If one end of a silk thread is fastened to a hook, and we pull the other end, it must be drawn straight before the force is transmitted to the hook. If we suppose the thread to have its direction changed by passing over a smooth peg, its parts severally become straight, and the force is transmitted undiminished to the hook.

(*c*) The characteristic property of a fluid is that it transmits force equally in all directions; this is true both of water, which is nearly incompressible, and of air, which is highly compressible.

(*d*) Besides the property mentioned in the last para-

graph, air enjoys this in addition—that if any quantity of it be enclosed and subjected to a varying pressure, its volume will vary inversely as the pressure. These properties of fluids will be more completely discussed further on in the volume.

The student must bear in mind that the above statements concerning bodies are only approximately true. In theoretical mechanics questions are solved on the supposition that bodies possess these properties because, in a very large number of cases, the solutions could only be obtained on these suppositions; and when thus obtained they are sensibly, or at all events approximately, correct. Still it is quite true that a perfectly rigid body, a perfectly flexible thread, a perfectly incompressible fluid, and a perfectly elastic gas, are abstractions, just as a line without breadth or thickness is an abstraction. Strictly speaking, all *solid* bodies are compressed, stretched, or distorted, when forces are applied to them, and sensibly so when the forces exceed a moderate amount; the most flexible body requires some force to bend it; *water* is in reality compressible; and when the pressure applied to a given quantity of *air* exceeds a certain amount, its volume sensibly ceases to be inversely proportional to the pressure. The various forms in which matter may exist suggest questions with which we are not here concerned; it is, however, well known that we may have solids as little rigid as lead or tallow, threads as little flexible as hempen ropes, liquids as little fluid as treacle or honey; and generally that matter may exist in almost every form intermediate to those forms which we have considered above as typical; and further that, at all events in some instances, simple substances may be made to pass through various states, from that of a highly rigid solid to that of a perfectly aëriform fluid, by mere change of temperature.

QUESTIONS.

1. Define the terms matter, body, point.

2. What is meant when it is asserted of two bodies that they contain equal quantities of matter? What is meant by the standard pound?

3. A lump of lead weighs $2\frac{1}{4}$ lbs., a lump of butter weighs 8 oz.; what ratio does the quantity of matter in the lead bear to the quantity of matter in the butter? *Ans.* 9 : 2.

4. When is a body said to be of uniform density?

5. Three cubic inches of mercury are found to weigh $1\frac{1}{2}$ lb., a cubic inch of cast iron $\frac{2}{7}$ lb.; compare the densities of cast iron and mercury. *Ans.* 4 : 7.

6. A gallon contains 277·274 cubic in.; calculate the weight of a gallon of distilled water at 60° F. *Ans.* 10·012 lbs.

7. What is meant by the specific gravity or specific density of a substance? State how the weight of a body of given size can be inferred when its specific gravity is known.

8. Assuming that a gallon of water weighs 10 lbs., what is the weight of a pint of mercury? What of a pint of petroleum? *Ans.* (1) 15·6 lbs. (2) 1·1 lbs.

9. A plate of glass 8 ft. long and 4 ft. wide weighs 100 lbs.; what is its thickness? *Ans.* $\frac{2}{9}$ in.

10. A yard of copper wire weighs half a pound; what is the diameter of the wire in inches? *Ans.* 0·236 in.

11. What is meant by a force? Give examples of forces.

12. What points must be known when a force is completely specified? What is meant by a *force* of one pound, i. e. the gravitation unit of force?

13. Give illustrations of the fact that the force exerted by gravity on a pound of matter is different at different places.

14. Illustrate the fact of the equality of action and reaction.

15. If bodies in contact are smooth, in what direction must their mutual action be exerted?

16. When one body is pressed against another, their surfaces being rough, what is meant by their angle of friction?

17. A piece of copper is urged against a surface of iron by certain forces; to what extent is the direction of the mutual action between the bodies known?

18. In the last question, suppose both bodies were greased, what modification would this introduce into the answer?

19. In Q. 17 and 18 what is the direction of the mutual action when the piece of copper is on the point of sliding?

20. What is meant by friction, and by the coefficient of friction?

21. A piece of copper is urged directly against a plane of iron by a force of 100 lbs.; under what circumstances will the whole friction be called into play?

22. State the laws of friction.

23. Define the terms *resultant* of two or more forces, and *components* of a force.

24. Give instances of a rigid body, a flexible body, an incompressible fluid, and a compressible fluid. What limitations are understood when these terms are applied to actual bodies? In what ways do these bodies (severally) transmit force? Give instances of states of matter intermediate to these typical forms.

N.B. It will frequently happen, both in the text and examples of the following pages, that numerical results are merely stated without being worked out; the student should in every case verify these statements, e.g. he should verify the statement as to the weight of a cubic foot and a cubic yard of water in Art. 4, p. 3, and that in Ex. 1, p. 4, that 24×7800 oz. are about 5 tons. He will find in subsequent chapters that the verification of many of these statements will bring to his notice points of considerable interest. He must also bear in mind that some of the answers to questions are exact, while others are approximate, e.g. the Answers to Q. 3, 5, 9 are exact, while those to Q. 6, 10 are approximate, and if given to five places of decimals would be 10·01232 and 0·23571 respectively. He will observe that in the latter case the nearest decimal of three places (0·236) is entered as the answer, not 0·235.

CHAPTER II.

CENTRE OF GRAVITY, AND EQUILIBRIUM OF TWO FORCES.

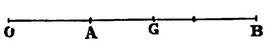
15. *Centre of gravity of two points.*—Let A and B be two points whose masses are m and n respectively; divide the line A B at G into two parts in the inverse ratio of the masses, i.e. find the point G from the proportion

$$AG : GB :: n : m.$$

The point G is called the *centre of gravity* (or the *centre of inertia*) of A and B.

Ex. 2.—Let the mass of A be 12 grains, and that of B 8 grains; the masses are in the ratio of 3 : 2, and if we divide A B into 5 equal parts,[1] and take A G, equal to two of them (and therefore G B equal to the remaining three), G will be the required centre of gravity.

FIG. 5.



If the distance of G from some point (O) in the line A B is required, it is easy to

[1] The division of a given straight line into any required number of equal parts may be effected by *Euclid*, vi. 9, 10; but in practice the following is a better method: 'A distance taken in the compasses by estimation, as nearly equal to one of the required parts as possible, is to be *stepped* along the line very lightly from one end; if the distance was by chance taken correctly, the line will be divided by it at once, but if not, the point of the compasses will not fall into the other extremity of the line, but will either exceed or fall short of the point, according as the estimated distance was taken greater or less than the true one. Without raising the instrument from the paper, this error is to be divided by the eye into as many parts as the whole line is required to be divided into, and the compasses opened or shut the quantity of one of these smaller divisions, according as the first assumed distance was too small or too large.' The trial must now be repeated until the resulting error is exceedingly small; this error can be *corrected by the eye.* See Bradley's *Practical Geometry*, p. 25.

see that the above proportion is equivalent to the equation

$$OG\,(m+n)=OA\,.\,m+OB\,.\,n,$$

which can be used with advantage in many cases.

If we suppose O to coincide with A, we see that

$$AG\,(m+n)=AB\,.\,n.$$

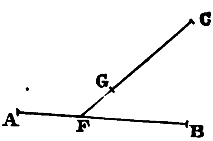
Ex. 3.—If A and B are 12 in. apart, and the masses of A and B are 5 oz. and 3 oz. respectively, we have

$$AG \times 8 = 12 \times 3\,;$$

therefore AG equals $4\frac{1}{2}$ in.

16. *Centre of gravity of three or more points.*—If A, B, and C are three points whose masses are severally m, n, and p, their centre of gravity can be found thus:—join any two of them, as A and B; find F the centre of gravity of A and B; join FC, and divide it in G in such a manner that

FIG. 6.



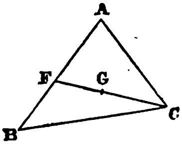
$$FG : GC :: p : m+n,$$

i.e. suppose the whole mass of A and B to be at F, and find the centre of gravity of F and C. This method is equally applicable whether A, B, C are in one straight line or not.

Ex. 4.—Let masses 3, 4, and 5 be placed at A, B, C, the angular points of a given triangle; find their centre of gravity.

Divide AB into seven equal parts, and take AF equal to four of them; F is the centre of gravity of A and B. Join FC; it is now as if we had a mass 7 at F and a mass 5 at C; divide FC in twelve equal parts, and take FG equal to five of them; G is the required centre of gravity. It is a useful exercise to combine the points in a different order, e.g. to begin with B and C; it will be found that the point thus obtained coincides with that given by the former construction.

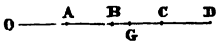
FIG. 7.



17. *Centre of gravity of several points in the same*

straight line.—Let O be a point of reference in the straight line; let the masses of A, B, C, D, be denoted by m_1, m_2, m_3, m_4, and their distances from O by x_1, x_2, x_3, x_4; then if G is their centre of gravity, it can be easily shown that

FIG. 8.



$$\text{OG}\,(m_1+m_2+m_3+m_4)=m_1x_1+m_2x_2+m_3x_3+m_4x_4,$$

and the like is true of any number of points.

Ex. 5.—Suppose there are four points, with masses 10, 8, 6, 4 oz. respectively, arranged at distances 1, 2, 3, 4 ft. from O, we shall have

$$\text{OG}\,(10+8+6+4)=10\times1+8\times2+6\times3+4\times4,$$

or, $$\text{OG}=2\tfrac{1}{7}\text{ ft.},$$

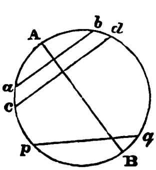
i. e. their centre of gravity is $2\frac{1}{7}$ ft. from O. The student should verify this by combining the points as in the last article; this can be done in several ways, but always with the same ultimate result.

18. *Centre of gravity of a body of given form.*—Since any body may be conceived to be made up of a number of points, it is possible to find its centre of gravity by an extension of the method above applied to several points; a few simple cases in which the bodies are assumed to be of uniform density will now be given:—

(*a*) A straight line is made up of a number of equally heavy points distributed uniformly along it; it is plain that their centre of gravity, and therefore that of the line, will be at its middle point.

(*b*) Any area may be conceived to be made up of a number of parallel straight lines, each having its centre of gravity at its middle point; if the area is such that these middle points range in a straight line, we then know that the centre of gravity must be somewhere in that line. Now suppose that by a similar process we can find a second straight line in which the centre of gravity must be; we then know that the centre of gravity, being in each of the two lines, must be at their point of intersection.

FIG. 9.



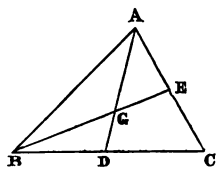
Thus, let the area be a circular plate, draw any chord $a\,b$; the plate may be conceived to be made up of a very large number of parallel straight lines, as $a\,b$, $c\,d$; the middle

points of all these chords will be in the diameter, A B, which bisects them at right angles. By reasoning in a similar way on a set of chords parallel to $p q$, we see that the centre of gravity must be in a second diameter, and consequently at the point of intersection of the two diameters, i. e. at the centre of the circle.

(*c*) By similar reasoning it may be shown that the centre of gravity of any parallelogram is at the point of intersection of the diagonals.

(*d*) It can be shown in like manner that the centre of gravity of a triangle can be thus found:—Let A B C be the triangle, D and E the middle points of B C and C A; draw A D and B E intersecting in G; this point is the centre of gravity of the triangle. It admits of an easy proof that D G is one third of A D.

FIG. 10.

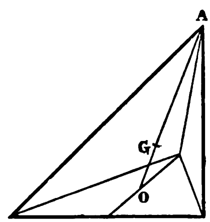


A similar method can be extended without much difficulty to a few solid bodies; on doing so the following results are obtained:—

(*e*) The centre of gravity of a prism or cylinder is at the middle point of its axis.

(*f*) The centre of gravity of a pyramid or cone is found thus:—Find O, the centre of gravity of the base, and draw a line from O to A, the vertex; take O G, one-fourth part of O A; G is the required centre of gravity.

FIG. 11.

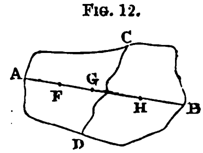


(*g*) The centre of gravity of the sphere is at its geometrical centre; and it may be noticed that the same is true of the circumference of a circle and of the surface of a sphere.

It must be borne in mind that the above rules are not generally true unless the bodies are of uniform density.

19. *Other cases of centre of gravity.*—The centres of gravity of many bodies may be found by the following method:—Let A B be any body divided into two parts by a boundary C D; let m and n denote the masses of A C D and B C D respectively, then $m+n$ is the mass of the whole body; let F and H be the centres of gravity of the parts, and G that of the whole body; these three points must be in one straight line which we will suppose to be A B.

FIG. 12.



Now we may treat the mass of A C D as if it were a heavy point at F, and that of B C D as if it were a heavy point at H; consequently we can find G by the proportion FG : GH :: n : m. Or, which is generally more convenient, we may use the equation

$$AG\,(m+n)=AF\,.\,m+AH\,.\,n.$$

It will be observed that in this equation there are five quantities, viz. m, n, A F, A G, A H; if any four are given, we find the fifth by the above equation.

Ex. 6.—The head of a hammer weighs 15 lbs. and its handle 5 lbs., the centre of gravity of the head (F) is 32 in. from A, the end of the handle, that of the handle (H) is 16 in. from A; find the position of (G) the centre of gravity of the hammer. On drawing a figure it will be seen that

$$AG \times 20 = AH \times 5 + AF \times 15;$$

or,
$$AG \times 20 = 16 \times 5 + 32 \times 15;$$

therefore,
$$AG = 28 \text{ in.}$$

Ex. 7.—Two balls (A and B) hang from the same point of the ceiling. A weighs 12 lbs. and B 7 lbs.; A is 1 ft. above the floor; where must B be placed that the centre of gravity (G) of the two may be 4 ft. above the floor?

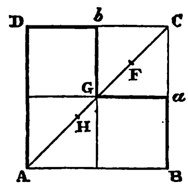
Since A G equals 3 ft., we must have

$$BG : 3 :: 12 : 7;$$

therefore B G equals $5\frac{1}{7}$ ft., or B is $9\frac{1}{7}$ ft. above the floor.

Ex. 8.—A square is divided into four equal squares, and one of them is taken away; find the centre of gravity of the remaining three squares.

FIG. 13.



A B C D the square divided as indicated; the centre of gravity of the whole square is at G, and that of the small square ($a\,b$) at F; hence the centre of gravity of the three remaining squares must be at some point, H, in the diagonal A C. Now the masses of the whole square and of the parts are proportional to the numbers 4, 3, and 1. Hence we have

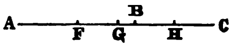
$$AG \times 4 = AF \times 1 + AH \times 3.$$

But A F equals $\frac{3}{4}$ A C, and A G equals $\frac{1}{2}$ A C; therefore A H must equal $\frac{5}{12}$ of A C.

Ex. 9.—The rod A C has a uniform section throughout; it consists of

two parts, AB and BC; the former is 5 ft. long and has a sp. gr. 6; the latter is 3 ft. long and has a sp. gr. 8; required the centre of gravity of the rod.

FIG. 14.



Let F and H be the centres of gravity of the parts, and G that of the whole rod; we have the mass of the parts proportional to 5×6 and 3×8, i.e. to 5 and 4; consequently,

$$AG \times (5+4) = AF \times 5 + AH \times 4.$$

Now AF equals $2\frac{1}{2}$ ft., and AH equals $6\frac{1}{2}$ ft.; therefore AG equals $4\frac{5}{18}$ ft.

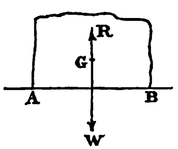
20. *Property of the centre of gravity.*—It may be assumed that when a body of moderate dimensions is near the earth's surface, the force of gravity acts on its parts along *parallel* lines; this is equivalent to assuming that vertical lines when near each other are parallel. This being the case, it admits of an easy proof that the resultant of the forces exerted by gravity on the parts of the body will act vertically downward *through the centre of gravity.* This is true when the body is turned in any way whatever; thus, in the case of a triangular board (fig. 10) the force of gravity on it will act vertically downward through G in whatever position the board is placed. This property is sometimes treated as a *definition* of the centre of gravity, and accordingly it is often said that the centre of gravity of a body is that point through which its weight always acts in whatever position it is placed.

21. *Condition of the equilibrium of two forces.*—When two forces act on a point, it is necessary and sufficient for equilibrium that they be equal and act along the same line in opposite directions. If, instead of acting on a point, they act on a rigid body, the same rule holds good. Simple as this rule is, we shall find that it includes a number of particular cases of interest and importance. It is worthy of remark that the rule is, strictly speaking, a result of experiment:—Suppose that P is a force of any kind

(e.g. the muscular force of an animal), and suppose it just sufficient to support a mass of M lbs. against gravity at a certain place. Let Q be another force (e.g. the elastic force of a metal spring) which would support an equal mass against gravity at the same place. The rule asserts that the force P would exactly balance Q if they acted in direct opposition.

22. *Body resting on a horizontal plane.*—Let G be the centre of gravity of the body which rests on the horizontal plane A B, under the action of gravity. Its weight (W) will act vertically downward through G, and will be balanced by the reaction of the plane (R), which must therefore be a force equal to W and acting vertically upward through G. There are several points connected with this question which call for notice.

FIG. 15.



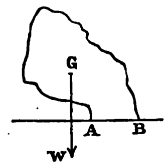
(*a*) The plane must be able to exert the reaction (R), in other words it must be strong enough to support the body. This point is always taken for granted unless the strength of the plane is in question. When solving questions in mechanics, the student should bear in mind that he is assuming the fixed planes, fixed points, &c., which enter the question to be sufficiently strong, and that this assumption, which he is apt to make without a second thought, suggests other mechanical questions of great difficulty and importance.

(*b*) The reaction is treated as a single force acting along a specified line. It must be borne in mind, however, that the body presses against the plane at many points; at each of these points a certain reaction is exerted the amount of which depends on the degree of compression which the body undergoes at that point. The data of the question are commonly such that there are no means of determining these forces severally; all that is known is

that they must be such as to have a resultant (R) acting as specified above.

(*c*) As all the reactions act upward, their resultant R must act through some point between A and B. If then the body is shaped in such a way that a vertical line through G passes outside the base (as shown in fig. 16) the reaction cannot act in a direction exactly opposite to the weight, and the body (unless supported) will not continue in the position shown. In other words, a body cannot rest without support on a horizontal plane unless a vertical line drawn through its centre of gravity cuts the plane within the base.

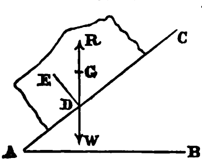
FIG. 16.



(*d*) If the body touches the plane at three, four, or more points, the base is the figure formed when the points are joined by straight lines; thus, a four-legged table touches the floor at only four points, and consequently the base is the rectangle formed by joining the points; if there were a fifth leg touching the floor within this rectangle it would not count, and similarly in other cases.

23. *Body resting on an inclined plane.*—Let G be the centre of gravity of the body which rests under the action of gravity on a plane A C inclined to the horizon A B. Draw the vertical line G D W; draw D E at right angles to A C. The only two forces acting are the weight of the body (W) and the reaction of the plane (R); they must therefore act in opposite directions. Now W acts vertically downward along G D, and consequently R must act upward along D G. This presupposes that two conditions are fulfilled: (*a*) D must not fall outside the base of the body; (*b*) the angle E D G must not exceed the angle

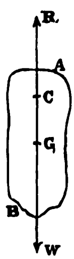
FIG. 17.



of friction. If the first condition were not fulfilled, the body without support would topple over; if the second, the body would slide down the plane. It will be observed that the angle E D G equals the angle C A B; in other words, the body will slide if the inclination of the plane to the horizon exceeds the angle of friction. If the plane were perfectly smooth the body would slide, unless the plane were exactly horizontal.

24. *Body suspended from a point.*—Let A B be the body suspended from a point C round which it can turn freely. Let G be its centre of gravity. The weight acts along a vertical line through G, and is balanced by the reaction of C; consequently, if the body is in such a position that the vertical line through G passes through C, the condition of equilibrium is fulfilled. This general statement includes two cases: (*a*) when G is vertically below C, (*b*) when G is vertically above C. These two cases differ from each other in an important point, which is the subject of the next article.

FIG. 18.



25. *Stable and unstable equilibrium.*—When a body is acted on by forces which fulfil the condition (or conditions) of equilibrium, and is in such a position that, if it receive any very slight displacement, the forces tend to bring it back to its original position, that position was one of *stable equilibrium*; thus, in the last article when G is *below* C the equilibrium is stable, for if the body were slightly displaced, it would swing back to its original position. If, however, the position of the body were such that on being slightly displaced the forces would tend to make the body move further from its original position, that position was one of *unstable* equilibrium; thus, in the last article, if G were vertically over C, the smallest displacement would cause the body to come into the

position shown in fig. 18. A third case is possible, viz. that when the body is slightly displaced, it has no tendency to leave its new position; its position in this case was one of *neutral* equilibrium; if, in the last article G were to coincide with C, the body would be in neutral equilibrium; for, let the body be turned in any direction, the reaction and the weight would be in equilibrium.

Ex. 10.—The student should consider the following case:—A sphere, whose centre is C, is loaded in such a way that its centre of gravity (G) is not at C; let A B be the diameter drawn through G. If the sphere is placed on a horizontal plane, the weight of the body and the reaction of the plane fulfil the condition of equilibrium when A B is vertical; the equilibrium is stable when G is below C, and unstable when G is above C. If G coincides with C, the equilibrium is neutral.

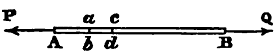
26. *Remarks.* (*a*) It must be noticed that stability admits of degrees, e.g. a box standing on its largest face is in stable equilibrium; it is also in stable equilibrium when standing on its smallest face; but it is clearly more stable in the former case than in the latter. It may easily happen that a body may be in a position which is, strictly speaking, one of stable equilibrium, and yet it would hardly stay at rest in it, e.g. a cylinder of wood thirty or forty inches long, standing on a base whose diameter is one inch, would be very liable to fall, though its position is, strictly speaking, one of stable equilibrium.

(*b*) A body may be in stable equilibrium with regard to displacements in some directions, and in unstable equilibrium in regard to displacements in other directions, e.g. a thin rectangle, such as a card, placed on its edge, is in stable equilibrium with reference to displacements in its own plane, and in unstable equilibrium with reference to displacements at right angles to its plane. As a matter of fact, a body left to itself in a position unstable in any direction will not stay in that position, e.g. a card placed on its edge will fall flat. We see, therefore, that when a body is in a position of unstable equilibrium it will not, practically speaking, stay in that position, though the forces acting on it fulfil the condition, or conditions, of equilibrium.

27. *Rod stretched by two forces.*—When forces are applied to a solid body and keep it in equilibrium there are three questions to which answers may be required: *first*, what relations must exist between the forces? *Secondly*, what alteration will they produce in the form of the body? *Thirdly*, under what circumstances will they break the

body? Our investigations in the present work will be for the most part limited to the first of these questions. The case, however, of a rod stretched by two forces admits of very simple treatment, and as the case is of great importance, we shall consider it from all three points of view. Let A B be the rod acted on by forces P and Q, tending to stretch it. We have already seen that they will be in equilibrium if they are equal and act in opposite directions along the line of the rod. These forces are transmitted through the rod from point to point, so that if any cross section $a\,b$ is considered, the forces tending to separate the rod at that point are the equal forces P and Q. The same is, of course, true of any other cross section $c\,d$. If, therefore, any small portion of the rod is considered, such as $a\,b\,c\,d$, we find that it is stretched by the two equal opposite forces P and Q, just as the whole rod is stretched by those forces.

FIG. 19.



The section of the rod has some definite magnitude, but it may be assumed that the greatest thickness, i.e. the greatest diameter of its cross section, is small when compared with the length of the rod. The forces are distributed over the cross section. In the neighbourhood of their points of application (A and B) the distribution will depend on the way in which they are applied; but, assuming the rod to be of uniform texture, at a moderate distance from A and B, the distribution will be uniform. A force thus distributed is called a *stress*, and is generally reckoned at so much per unit of area, e.g. in pounds per square inch. Thus, if P were a force of 5000 lbs., and the area of the cross section 4 sq. in., the stress is 1250 lbs. per sq. inch.

The forces P and Q may have any physical origin whatever, e.g. P may be a weight fastened to the end A;

the end B may be formed into a loop or eye, and the rod may hang suspended by that eye from a fixed point; the reaction of this fixed point is the second force Q. In such a case the rod is frequently said to be stretched by the force P; but though there is no objection to this way of speaking, the student must bear in mind that two forces at least are necessary to keep the rod in equilibrium.

28. *Elasticity.*—In considering the second point we come upon one of the most important properties of bodies viz. *elasticity.* When force is applied to a solid body, it produces a change of form or volume or both. If we suppose the force not to exceed a certain amount, the body tends to recover its original condition, and when the force is withdrawn, actually resumes its original form and volume. The tendency of the body to recover its original condition is called *elasticity.* When two forces are applied to a rod, as in the last article, they lengthen it, though ordinarily by a very small amount; and, if the forces do not exceed a certain magnitude, the rod returns to its original length, when they cease to act.

In the case we have supposed the elongation (l) is proportional to the length (L) of the rod, and to the tensile stress, i. e. to the force estimated per unit of area of the cross section. If then P denotes one of the stretching forces and A the area of the cross section, we must have l proportional to L and $P \div A$, or

$$l = \frac{P\,L}{E\,A},$$

where E denotes a number depending on the material of the rod, which is called *the modulus of elasticity* for that material. It is commonly estimated in pounds per square inch; when this is done, P must be reckoned in pounds, A

in square inches, and then L and l may be in any units, i.e. both in inches or both in feet, &c. The mean numerical values of E for different materials have been ascertained by careful experiment. Some of the values are given in the following table:—

TABLE II.

MODULI OF ELASTICITY.[1]

	lbs. per sq. in.		lbs. per sq. in.
Cast steel	31,000,000	Teak	2,400,000
Wrought iron	28,450,000	Red pine	1,840,000
Cast iron	19,550,000	English oak	1,450,000
Drawn copper	17,870,000	Beech	1,350,000
Drawn brass	15,580,000	Riga fir	1,330,000
Flint glass	8,325,000	Elm	700,000

Ex. 11.—A wrought-iron rod 50 ft. long and 3 sq. in. in section is stretched by a force of 6 tons; find the elongation.

Here the values of E, A, P, L, in the above formula are given, viz. 28,450,000 lbs. per sq. in., 3 sq. in. 13,440 lbs. and 50 ft. Consequently the elongation in feet is

$$\frac{50 \times 13.440}{3 \times 28,450,000},$$

i.e. between $\frac{1}{10}$th and $\frac{1}{11}$th of an inch.

Ex. 12.—A rod of wrought iron 10 ft. long and 1 sq. in. in section, when stretched by a force of 1262 lbs., was found to have lengthened by 0·0052 of an inch. What value of the modulus can be deduced from this experiment?

Since $\text{A E} l = \text{P L}$, and P, A, L, l are respectively 1262 lbs., 1 sq. in., 120 in., and 0·0052 in., we find $\text{E} \times 0{\cdot}0052 = 1262 \times 120$, and therefore E must equal 29,120,000 lbs. per sq. in.

Ex. 13. A copper wire hangs vertically, being suspended from one end; it is found that near the point of suspension a portion of the wire, which when unstretched was 1 ft. long, is now 1·001 ft. long; what is the length of the wire?

If x is the length in inches of the part which stretches the portion under consideration, and A the area of the cross section in square inches, the volume of the copper whose weight produces the elongation is x A cubic in.,

[1] The first column of the above table is taken (with a change of units) from Prof. Everett's Table, *Proceedings R. S.*, vol. xvi. p. 248. The second column from Barlow's *Strength of Materials*, p. 82.

or $x A \div 1728$ cubic ft., and its weight (Table I.) is $550\ x A \div 1728$ lbs. Hence we obtain the equation

$$\frac{1 \times 550\ x A \div 1728}{A \times 17{,}870{,}000} = \frac{1}{1000},$$

which gives $x = 4679$ ft.

29. *Remarks on the modulus of elasticity.*—The numbers entered in Table II. are, it must be borne in mind, mean numerical values, and consequently experiment with different specimens of any given substance must be expected to give results varying more or less from the registered value. An instance of this is given in Ex. 12, where the data are those of an actual experiment. The variations from the mean are apt to be large in the case of different kinds of wood. To put the fact of the variation and its amount in a clearer light, we will give a few particulars respecting wrought iron and steel. There are two ways in which the modulus can be experimentally determined:—(1) by placing a bar of the material horizontally on two points and loading it in the middle, and then measuring the amount of its deflection; (2) by exposing the rod to tension and measuring the elongation actually produced. The modulus of wrought iron registered in Table II. was obtained by the former method. The modulus has also been obtained by the second method, and with this result—in the case of eleven different specimens of wrought iron the modulus was found to range from 31,976,920 to 26,761,800 lbs. per square inch.[1] In the case of five different kinds of steel, experiment by direct tension gave values of the modulus varying from 29,918,320 to 31,359,340 lbs. per square inch.[2] These results closely agree with those obtained by Mr. Fairbairn, who determined the modulus for

[1] Specimens 10–20, Table ix. p. 146, of *A Treatise on the Elasticity, Extensibility, and Tensile Strength of Iron and Steel*, by Knut Styffe.

[2] Ibid. Table ix.

steel bars by bending, and in forty-eight different specimens of steel found its value to range between 33,446,000 and 28,353,000 lbs. per square inch. In four specimens, however, the modulus had a much smaller value.[1]

30. *The limit of elasticity.*—In Article 28 the statement of the law of elasticity was limited by the condition that the stress should 'not exceed a certain amount.' This certain amount is the *elastic limit*, and is a point that has been a good deal discussed. The simplest view of the question amounts to this:—Conceive a rod of steel exposed to the action of a gradually increasing tensile stress; the elongations are at first to each other as the stresses which produce them; so long as this is the case the stress is within the elastic limit; but when the stresses exceed a certain amount, the elongations increase more rapidly than in proportion to the stresses. The stress at which this change in the proportions definitely begins is the elastic limit. Any stress decidedly in excess of the elastic limit would impair the elasticity of the bar. Thus, Mr. Fairbairn[2] found that the average elastic limit in the case of nine steel bars was 12,136 lbs. per square inch, the greatest being 15,586, and the least 11,581 lbs. per square inch, i.e. up to between 5 and 7 tons per square inch the elongations were proportional to the stress. For stresses in excess of this the elongations were greater than in proportion to the stress, and by such stresses the elasticity of the rod was decidedly weakened.

31. *Tenacity.*—When a rod or bar of a given substance is exposed to the action of a gradually increasing tensile stress, a limit is at last reached at which the stress cannot be supported, and the rod is torn asunder; the tensile stress, reckoned in pounds per square inch of the original cross section of the rod, which, when gradually

[1] Fairbairn on *Iron*, Table i.

[2] *Report* of British Association for 1867 or *Treatise on Iron*, Table i.

applied, just tears the rod, is called the tenacity of the substance. In different specimens of the same substance the value of the tenacity is liable to considerable variation; consequently the numerical values of the tenacity given in the subjoined table must be taken as mean values from which the actual values in various specimens may differ very considerably.

TABLE III.

TENACITIES.

	lbs per sq. in.		lbs. per sq. in.
Steel	115,000	English oak	17,500
Wrought iron (bars)	60,000	Elm	13,500
Iron (wire ropes)	90,000	Riga fir	12,000
Copper wire	60,000	Larch	10,000
Cast brass	18,000	Hempen rope	5,500
Cast iron	16,000		

In illustration of the variations in the value of the tenacity of different specimens of the same substance, the following cases may be quoted:—In nine specimens of steel the average tenacity was found to be 90,379 lbs. per square inch, the extreme values being 116,183 and 59,538 lbs. per square inch.[1] The former value is by no means excessively large, and by different processes of hardening tenacities of 168,530, 173,951, and even 195,018 lbs. per square inch of the original area have been imparted to steel.[2] Again, in the case of cast iron the mean of nine determinations of the tenacity was 16,720, the extremes being 21,907 and 13,434 lbs. per square inch.[3] And similarly in other cases.

32. *Working stress.*—In practice rods are never subjected to stresses nearly equal to the tenacity or ultimate strength; as a rough general rule, they should not be permanently exposed to a stress exceeding $\frac{1}{8}$th or $\frac{1}{10}$th

[1] Fairbairn's Table, Treatise on *Iron*, Table ii.

[2] Knut Styffe, p. 136.

[3] Fairbairn on *Iron*, p. 217.

of their ultimate strength; such a stress is called the working stress. The reason for fixing on this particular proportion becomes evident on considering the facts mentioned above. In the nine specimens of steel mentioned in Article 30 the average elastic limit was 12,136 lbs. per square inch, while the average tenacity of nine specimens of the same kind was 90,379 lbs. per square inch; we see, therefore, that a stress of $\frac{1}{8}$th or $\frac{1}{10}$th of the tenacity is well within the limits of elasticity. This rule applies very generally to other cases besides that of rods exposed to tensile stress.

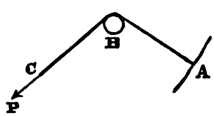
33. *Set or permanent elongation.*—The condition of a rod when exposed to stresses intermediate to the elastic limit and the tenacity demands notice. Confining our remarks to a rod or bar of iron, the chief point is this—the effect of the stress is not merely to stretch the rod more than it would have been stretched had its elasticity continued perfect, but when the stress is withdrawn the rod is found to have been permanently lengthened, or in other words to have undergone a 'set.' The mere fact of a small set having been produced does not prove that the elasticity has been injured, and it may happen that stresses less than the elastic limit may produce small sets.[1] Beyond that limit, however, as the stresses increase the magnitudes of the sets increase, and in a continually increasing proportion. Thus, in a very carefully executed experiment on an iron rod, until the stress reached 25,952 lbs. per square inch, no permanent elongation was produced, and then it amounted to 1–250,000th of the length of the rod; when the stress reached 33,733 lbs. per square inch, the rod had been permanently lengthened by 1–2800th of its length. From this point the sets rapidly increased until at 56,488 lbs. per square inch the

[1] See Exp. iv. p. 169, Exp. xvi. p. 181, &c., of Mr. Fairbairn's Report on *Steel, British Association* for 1867.

rod broke, after having been lengthened by nearly 1–5th of its original length.[1] Similar phenomena are presented by other substances, such as steel; but in the case of brittle substances, such as cast iron, there is little or no permanent elongation before the stress equals the tenacity.

34. *Case of a flexible thread.*—Before leaving the subject of tensile stress it will be well to notice the case of a thread or rope as distinguished from a rod. A thread like a rod can transmit a force tending to stretch it, but is incapable of transmitting a force tending to compress it—it can transmit a *pull*, but not a *thrust*. Moreover it will transmit a tensile force in the direction of its length, even when that direction is changed by passing over a fixed point. Let a thread A B C be fastened to A, and pass over a fixed point B; let it be pulled by a force P applied to the end C; the effects of P are, that the parts A B and B C will be straight, and that P will exert on A a force in the direction A to B. If we suppose the thread to be perfectly flexible, and the point B to be perfectly smooth, the force P is transmitted without diminution to A. In the present work we shall always make these suppositions, but it will be well, for the information of the student, to make two remarks. (1) If we suppose the thread to be a strong but fine silk thread, and the point B to be a small pulley on a fine steel axle, the above conditions are very nearly realised. (2) Even if we suppose the thread to be flexible, if there is a moderate degree of roughness between the fixed point and the thread, only a part of P is transmitted to A. The student may easily convince himself of this by observing that when a weight is raised by means of a rope passing over a pulley, much less force is needed if the pulley turns

FIG. 20.



[1] Knut Styffe on *Iron and Steel*, p. 134, and plate V.

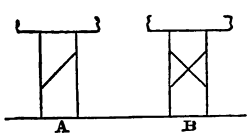
than if the axis were jammed and the pulley kept from turning.

35. *Compression.*—When a body is placed on a fixed plane or table and a heavy weight is placed upon it, it is under the action of two forces, or rather of two stresses i.e. distributed forces, viz. the weight acting downward and the reaction of the fixed table acting upwards. We have here a case of the equilibrium of two forces, though it is very usual to speak of the body as compressed by one force, viz. by the weight, and the body is sometimes said to sustain a *thrust* of the same amount. When a solid undergoes a small compression, it resists with a force proportional to the compression, and when the compressing force is withdrawn, it recovers its size and shape. In short, the same law of elasticity holds good as in the case of small extensions. When the compressing force is large, the body undergoes a permanent change of shape; and when the force is still further increased, the body completely yields or is crushed. The way in which the body yields depends upon its texture, and may be either of the five following:—[1]

(*a*) *Crushing by splitting*, when the substance divides in a direction nearly parallel to that of the pressure; this takes place in hard homogeneous substances of a glassy texture.

(*b*) *Crushing by shearing*, when the substance divides along a plane inclined at a certain angle to the direction of the force, the upper part of the substance sliding upon the lower; this takes place in substances of a granular texture such as cast iron, and most kinds of stone and brick. Thus, if A and B are two blocks of cast iron whose heights are about twice the thickness

FIG. 21.



[1] Rankine, *Applied Mechanics*, p. 302.

of their bases, the fracture will take place along one inclined plane, as in A, or along two, as in B. The inclination of the planes of fracture to the direction of the crushing force is about 40°.

(*c*) *Crushing by bulging*, in which the sides of the block swell out, as in the case of ductile and tough materials such as wrought iron. It also characterises steel, though this metal sometimes cracks as well as bulges.

(*d*) *Crushing by buckling or crippling* occurs in fibrous substances under the action of a thrust along the fibre; it consists in a lateral bending and wrinkling of the fibres, which are sometimes split asunder; this takes place in timber, in plates of wrought iron, and in bars longer than those which give way by bulging.

(*e*) *Crushing by cross breaking* takes place in long columns or struts whose length is from ten to twenty times the diameter of the cross section; in these cases the substance bends in the middle and the fracture is produced by cross breaking in the same way as an overloaded lever is broken.

The following table gives an average value for the stress in pounds per square inch for which short blocks of the several materials yield by crushing. It may be assumed that a substance should not be exposed to the long or oft-repeated action of a thrust exceeding 1-10th of its ultimate strength.

TABLE IV.

RESISTANCE TO COMPRESSION.

	pounds per sq. in.		pounds per sq. in.
Steel	225,000	Dry oak	9,000
Cast iron	110,000	Riga fir	6,000
Wrought iron	36,000	Granite	8,000
Cast brass	10,000	Sandstone	4,000
Elm	10,000	Brick	800

QUESTIONS.

1. Give two methods of finding the centre of gravity of two heavy points.

2. If a weight of 12 lbs. is 4 ft. from one end of a rod, and a weight of 18 lbs. is 10 ft. from that end, their centre of gravity is 7·6 ft. from that end. Show that this result can be arrived at by both methods.

3. Give a method of finding the centre of gravity of three or more heavy points.

4. Draw a square and mark its corners in succession A, B, C, D; let a weight of 3 oz. be placed at A, one of 4·5 oz. at B, one of 7·5 oz. at C, and one of 6 oz. at D. Find the centre of gravity of the four weights. Work this out by construction in at least two different ways, and show that the same result is obtained in both ways.

5. In the last question show where a weight of 10·5 oz. must be placed so as to bring the centre of gravity of the five weights to the intersection of the diagonals of the square.

6. Give a formula for finding the centre of gravity of any number of points ranging in a straight line.

7. AB is a rod 10 ft. long, a weight of 3 lbs. is placed at A, one of 5 lbs. four feet from A, one of 8 lbs. at B; show that the centre of gravity of the three weights is $6\frac{1}{4}$ ft. from A.

8. In the last question where must a weight of 4 lbs. be placed to bring the centre of gravity of the four weights to the middle of the rod?

Ans. At A.

9. State the rules for finding the centre of gravity of a straight line, a circle, a parallelogram, a triangle, a prism, a cylinder, a pyramid, a cone, a sphere, all the bodies being of uniform density.

10. In Ex. 8 (p. 20) show that the result may be obtained by treating the body as if made up of three heavy points at the centres of the three small squares respectively.

11. A circular plate of wood or metal has cut in it a circular hole of given size and position; show how to find its centre of gravity.

12. In the last question, if the radius of the hole is $\frac{1}{3}$rd of the radius of the plate, and if the centres of plate and hole are 18 in. apart, show that the centre of gravity is $2\frac{1}{4}$ in. from the centre of the plate.

13. A wire is bent into the form of a triangle whose sides are 6, 10, and 8 in.; find its centre of gravity.

14. A cross is made up of six equal squares; find its centre of gravity.

15. Two balls of equal radius are of cast iron and copper; they are

placed in contact; show that their centre of gravity—which is in the line joining their centres—is at a distance of $\frac{9}{10}$ths of the radius from the centre of the copper ball.

16. State the mechanical property of the centre of gravity, which is often called a definition of the centre of gravity. On what supposition as to the action of gravity does this point enjoy this property?

17. State the condition of the equilibrium of two forces acting on a point or rigid body. Show by an illustration that this rule is not a mere truism.

18. State the condition to be fulfilled when a body under the action of gravity will rest:—(1) on a horizontal plane; (2) on a rough inclined plane; (3) when suspended from a point round which it can turn freely.

19. A parallelogram[1] a foot wide and a foot in height will just stand on a horizontal plane; show that its angles must equal 45° and 135°.

20. A rhombus in a vertical plane will stand with one edge on a horizontal plane, whatever be the size of its angles.

21. A rectangular block of stone, whose base is a foot wide, rests on an inclined plane also of stone; it is found to be on the point of sliding; why must the inclination of the plane to the horizon be 33°? If the block is also just on the point of toppling over, why must its height be 1·54 ft.?

22. A rectangular block put on an inclined plane just falls over when the plane is inclined at an angle of 10°; if its base is 2 ft. wide, what is its height? *Ans.* 11·34 ft.

23. If a piece of iron is made into the form of a brick, and placed on an inclined plane, it is found—due precaution being taken against accidental adhesion—that it just slides down when the plane is at a certain inclination, whichever face it rests on; what law of friction does this fact illustrate?

24. A triangular board weighs 4 lbs., a weight of 2 lbs. is placed at one corner; if the board is suspended by another corner, find the position in which it comes to rest.

25. In the last question, if the weight of 2 lbs. were hung by a thread from the corner, instead of being fastened to the corner, show that the position of the triangle would be unaffected.

26. Two balls weighing 4 lbs. and 1 lb. have their centres joined by a

[1] In this question the parallelogram may be supposed to be the end of a right prism presented endwise to the plane of the paper; or, which comes to the same thing, the equilibrium of the body with regard to other planes besides the plane of the paper is not to be considered. A similar remark applies to many of the *examples* and questions.

rod without weight 50 in. long; if they are suspended by a point 8 in. from the centre of the larger ball, in what position will they come to rest?

27. State the distinction between stable, unstable, and neutral equilibrium, giving an example of each.

28. A rectangular block is placed on the ground; which are its positions of stable and which of unstable equilibrium? In which position is its equilibrium most and in which least stable?

29. In Question 26 have you considered whether your answer gives a position of stable or of unstable equilibrium? Why ought you to consider that the second position is excluded by the terms of the question?

30. Why does an egg-shaped body (unless weighted) rest on a table with its long axis horizontal?

31. A rod is stretched by two forces; how are the forces transmitted through the body? What is a stress?

32. A rod with a cross section of $\frac{1}{3}$rd square in. is suspended by one end, and has a weight of one ton fastened to the other; what is the tensile stress at any cross section? What are the two forces which stretch this rod?

33. Define the modulus of elasticity. When a rod is stretched, what is the relation between the original length, cross section, elongation, and modulus?

34. A copper wire $\frac{1}{20}$th in. in diameter sustains a weight of 20 lbs. What will be the length of a portion of the wire which, when unstretched, was a yard long? *Ans.* 3·00171 ft.

35. A bar of iron of uniform cross section hangs suspended by one end; it is known that one foot of its length is lengthened by 0·001 ft. What length of the rod must hang below that foot? *Ans.* 8,404 ft.

36. In what sense can it be stated that the modulus of elasticity of a given material—such as wrought iron, copper wire, &c.—is a constant number?

37. What is meant by the elastic limit of a given material? Describe briefly the behaviour of a steel bar under a gradually increasing tensile stress. What is (about) the elastic limit of a steel bar?

38. What is meant by the tenacity of a substance? Can the tenacity of a given substance be assigned with any great exactness without actual trial? Contrast in this respect the tabular value of the tenacity of steel with that of the modulus of elasticity of steel. What is meant by a working stress?

39. A copper wire is fastened to a weight of 200 lbs. resting on the floor; an attempt is made to lift the weight by means of the wire (without

any jerking); the weight is just raised off the floor, and then the wire breaks; what may be presumed to be the diameter of the wire? What weight ought such a wire to bear safely? If the wire were exposed to the greatest weight it could safely bear, what would be its elongation per yard? *Ans.* (1) 0·06515 in. (2) 20 lbs. (3) 0·0121 in.

40. A weight of one ton has to be lifted by a hempen rope; what ought to be the circumference of the rope? *Ans.* $7\frac{1}{8}$ in.

41. What is meant by a 'set?' Give an example of the behaviour of an iron rod under a gradually increasing tensile stress.

42. Describe briefly how a force is transmitted along a flexible thread.

43. When a small prism of any substance is placed on a table, and a weight, say of 100 lbs., is put upon it, what forces act on the prism?

44. Describe briefly the various ways in which a body may yield to a compressing force.

45. A rod of cast iron 10 ft. long, and 1 sq. in. in section, when compressed by a force of 2100 lbs., was shortened by 0·01875 in. What was its modulus of elasticity? *Ans.* 13,440,000 lbs. per sq. in.

46. Show that the greatest height to which a brick wall can be carried with safety, so far as the power of brick to resist crushing is concerned, is about 92 ft. How is it that brick structures are often raised to a much greater height than this?

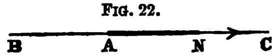
N.B. In several of the above questions (e.g. Q. 4, 5, 13, &c.) it is intended that the student should obtain the solution by drawing the proper diagrams to scale; performing the required operations with compasses, scale and protractor, in a way like that indicated in Ex. 4, p. 17. It is, however, in the following chapter that the student will have most occasion to use constructions, and he should do this in all cases where the need of it is indicated either in the text or examples, or in the questions at the end. He should use as large a scale as his paper will allow; the numerical results thus obtained ought to agree closely with those given in the book, but exact agreement is not to be expected. The results given in the book have been obtained by calculation, and are presumed to be correct.

CHAPTER III.

EQUILIBRIUM OF THREE OR MORE FORCES.

36. *Representation of forces by lines.*—Let a force act on a point A along a line B C in the direction B to C, and let it contain any given number of units of force; from A towards C set off a line A N containing as many units of length as the force contains units of force—e.g. if the force is one of 5 lbs., and we use as a unit of length half a quarter of an inch, then we must set off A N equal to $\frac{5}{8}$ths of an inch. The line A N will completely represent the force. For (see Art. 8) a force is completely specified when we know (*a*) the point on which it acts (A), (*b*) the line along which it acts (B C), (*c*) the direction of its action along the line (towards C), (*d*) the number of units of force it contains. Now A N is set off from A, along B C, towards C, and contains as many units of length as the force contains units of force. It is plain, therefore, that A N represents the force in every respect. The student will be kept from making many mistakes if, when representing forces in a diagram, he indicates by an arrow-head the direction in which he supposes each force to act.

FIG. 22.



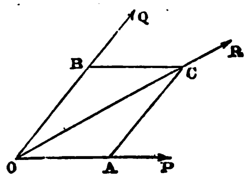
37. *Parallelogram of forces.*—When two forces act upon a point their resultant can be found by the following construction:—Draw two lines to *represent* the forces (as explained in the last article), using the same scale for both, and with these lines as sides construct a parallelogram; draw the diagonal of the parallelogram which passes *through* the given point; that diagonal *represents* the

resultant (Art. 13). The reason of this rule will be given in a subsequent chapter. For the present we shall assume it to be true as a matter of fact.

Ex. 14.—Let a force (P) of 4 lbs. act on O from O to P, and a force (Q) of 5 lbs. from O to Q; let POQ be an angle of 50°; find the resultant of P and Q.

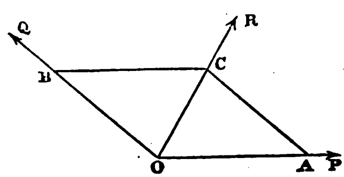
Set off OA containing four units of length (say $\frac{4}{8}$ths of an inch), and OB, containing $\frac{5}{8}$ths of an inch; complete the parallelogram OACB; join OC; this line represents the resultant (R). It will be found by measurement that OC is $8\frac{1}{5}$-eighths of an inch long, and that AOC and BOC are angles of 28° and 22° respectively. Consequently the resultant R is a force of $8\frac{1}{5}$ lbs., and acts as shown in the figure. The student should verify these results. If a large scale is used, or if the results are obtained by calculation, it will be found that the resultant is a force of 8·17 lbs., and that POR and ROQ are angles of 27° 58′ and 22° 2′ respectively.

Fig. 23.



38. *Components of a force.*—If a force R acts on O along the line OR (fig. 24) and is represented by OC, and if through O we draw any two lines OP, OQ, and complete the parallelogram OACB, the forces represented by OA and OB are equivalent to R, and can be used to replace R. These forces are called the *components* of R (Art. 13). It will be observed that two forces acting on a point can have only one resultant; but one force acting on a point can have any number of pairs of components acting on that point. If the lines OP and OQ are at right angles to each other, the components of R along these lines are called *rectangular* components of R.

Fig. 24.



Ex. 15.—Let OP, OQ, OR, be three lines drawn through O, and let POR and ROQ be angles of 62° and 78°; a force of 7 units acts from O to R; find its components along OP and OQ.

Take OC, containing 7 units of length, complete the parallelogram OACB, then OA and OB represent the components in question, which are forces of 10·65 and 9·62 units respectively. The student should verify this by a construction drawn carefully to scale. It will be observed that each of the components separately is greater than the given force; this is due to the fact that the components act in such a manner as to a great extent to neutralise each other, so that *jointly* they are only equivalent to a force of 7 units.

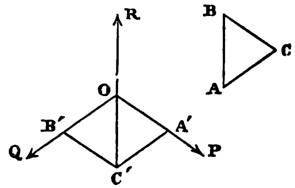
Remark.—The particular units both of length and of force employed in the last article are quite arbitrary. All that is necessary being that in any given question the same unit of length be used throughout the solution, and likewise the same unit of force. It is often convenient not to use any particular units, but to express the results in ratios, thus:—In Art. 38 we may say that P : Q :: OA : OB, and then OA : OC :: P : R; in like manner, in Ex. 15 we might take OC of any length, and then, on completing the parallelogram, we should have OC : OA :: R : the component along OP, and similarly OC : OB :: R : the component along OQ—e.g. if OC were 2 in. long, OA would be found to be about 3·04 in. long, and then 2 : 3·04 :: 7 : 10·64 the force required.

39. *Equilibrium of three forces acting on a point.*—If the resultant of any two of the forces be found by Art. 37, the three are reduced to two forces, and these will be in equilibrium if they are equal and act in opposite directions along the same line (Art. 21). Consequently the condition of equilibrium of three forces acting on a point is that any one of the forces be equal and opposite to the resultant of the other two, as determined by the parallelogram of forces. Thus, in Ex. 14 the two forces of 4 lbs. and 5 lbs. acting on O, from O to P and O to Q, would be balanced by a force of 8·17 lbs. acting on O from C to O. It is a very common mistake to confuse the resultant of two forces with the force that balances them; the one force is equal and opposite to the other.

40. *The triangle of forces.*—The condition of equilibrium of three forces admits of being stated in a somewhat different form, commonly called the 'triangle of forces,' as follows:—Let P, Q, R be three forces which keep the point O in equilibrium. Draw any three lines parallel

to O P, O Q, O R respectively, so as to form a triangle A B C. The forces acting on O will be in equilibrium if two conditions are fulfilled, viz. (1) supposing P to act in the direction B to C, Q must act in the direction C to A and R in the direction A to B; (2) the forces must be proportional to the sides of the triangle to which their lines of action are severally parallel; i.e. we must have

FIG. 25.



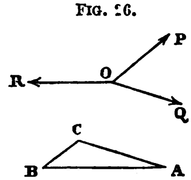
$$P : Q :: BC : CA \text{ and } Q : R :: CA : AB.$$

This is evidently true, for if O A′ be taken equal to B C, and the parallelogram O A′ B′ C′ be completed, the triangle A′ O C′ is in all respects equal to C B A, and consequently O C′, which represents the resultant of P and Q, represents a force equal and opposite to R; so that the forces P, Q, R, acting in the way supposed, are in equilibrium.

Ex. 16.—Forces of 15 and 20 units act in directions containing an angle of 120°; find the force which balances them.

Draw P O Q an angle of 120°, in O P take any length O A (e.g. 3 in.), and in O Q take O B equal to $\frac{4}{3}$rds of O A (4 in.) O A and O B represent forces of 15 and 20 units; complete the parallelogram O A C B, and join O C, the length of which will be found to be 1·2 times O A (3·6 in.) and consequently the required force (R) is one of 18 units; the angles Q O R and R O P will be found to equal 133° 54′ and 106° 6′. The student should verify these results by actually making the construction.

FIG. 26.



Ex. 17.—Show how to arrange three forces of 4, 10, and 13 units, that thay may be in equilibrium when acting on a point.

Draw a triangle A B C whose sides are in the proportion of 4 : 10 : 13 (e.g. 4, 10, and 13 quarter inches respectively). Take any point O, and draw O P, O Q, O R, respectively, parallel to B C, C A, A B; the forces of 4, 10, and 13 units act along these lines respectively. The student must observe that if he

supposes the force P to act on O in the direction B to C, he must make Q act in the direction C to A, and R in the direction A to B.[1] The student should make this construction, and also examine the effect of supposing that P acts on O in the direction O to B. He will find in both cases that the angles POQ, QOR, and ROP, are respectively 48° 30′, 166° 40′, and 144° 50′. These angles are, of course, the supplements of the angles of the triangle ABC.

41. *Three forces in equilibrium when acting on a rigid body.*—In this case the lines along which the forces act must, when produced, pass through a common point. It is not necessary that this point be a part of the body; it may in fact be situated at any distance from the body. When this preliminary condition is fulfilled, it remains that the forces also fulfil the conditions of equilibrium of three forces acting on a point (Art. 39, 40). Thus, if P, Q, R, are three forces acting on a rigid body at the points A, B, C respectively; the first condition of their equilibrium is that the three lines AP, BQ, CR, shall, on being produced if necessary, pass through a common point O; the second condition is that they shall be in equilibrium if the forces are supposed to act at the point O along the lines OP, OQ, OR. It may be noticed that these conditions cannot be fulfilled unless the lines along which the forces act are in one plane.

FIG. 27.

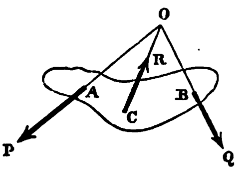
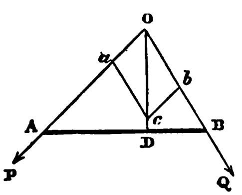


FIG. 28.



Ex. 18.—AB is a rigid rod acted on by two forces (P and Q) of 10 and 15 units respectively; the angles PAB and QBA are 135° and 120° respectively; find the force which must act on the rod to balance them.

Produce PA and QB to meet in O; take Oa, Ob, of 10 and 15 units

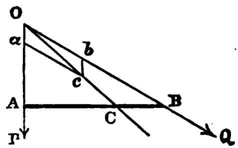
---

[1] The student must bear in mind the distinction between the *direction* of a force and its *line of action* (see Art. 8).

respectively (e.g. 1 and $1\frac{1}{2}$ in.); complete the parallelogram O acb; join O c and produce it to meet A B in D; on measuring O c it will be found that the resultant of P and Q is a force of 20·07 units. The force which balances P and Q must therefore be one of 20·07 units, and must act through D along the line D O in the direction D to O. It will be found that A D is (about) $\frac{13}{20}$ths of A B, and the angle A D O is 88° 47′. If we suppose D to be a fixed point or *fulcrum,* round which the rod is capable of turning, then the force we have just determined is the reaction of the fulcrum against the rod. The pressure produced by P and Q on the fulcrum is, of course, exactly equal and opposite to the reaction.

Ex. 19.—Let A B be a rod capable of turning round a fixed point or fulcrum C; it is acted on at A and B by forces P and Q in known directions O A, O B; given the force P, find the force Q and the pressure on the fulcrum.

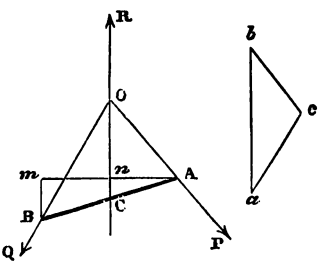
FIG. 29.



Join O C; take O a containing P units of length, draw ac and cb parallel to O B and O A, then O b and O c will serve to determine Q and the pressure on the fulcrum; e.g. if A C is $\frac{2}{3}$rds of A B, O A B a right angle, O B A an angle of 30°, and P a force of 50 units, it will be found that Q must be a force of 200 units, and that the pressure on the fulcrum is a force of 229 units.

Ex. 20.—Forces of 8, 10, and 15 units act respectively at the ends and middle point of a straight rod; find their directions when they are in equilibrium.

FIG. 30.

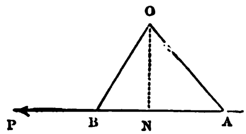


Draw a triangle abc whose sides are proportional to 8, 10, 15. Take any point O, and draw O P, O Q, O R, parallel to bc, ca, and ab respectively; forces P, Q, R, of 8, 10, and 15 units, would balance if they acted on the point O. In O P take any point A, draw A n m at right angles to R O produced, and make mn equal to n A; draw m B at right angles to A m, cutting O Q in B; join A B, cutting R O produced in C. It is plain that A C equals C B, and consequently the forces must act on the rod along the lines A P, B Q, and C R, as shown in the figure.

42. *Moment of a force.*—Let P be any force acting along the line A B and O any point; from O let fall a

perpendicular O N, on A B. The number of units of length in O N multiplied by the number of units of force in P—

FIG. 31.



or, as it may be more briefly stated, the product of the line O N and the force P—is called the moment of P with respect to the point O. If the line A B be taken to represent the force P, the product of the lines O N and A B is twice the area of the triangle A O B. So that the moment of P, with respect to O, is represented by twice the area of the triangle A O B. And when the moments of two forces with respect to the point O are under consideration, they are in the same ratio as the areas of two such triangles. Suppose the body on which P acts to be capable of turning round a line or axis passing through O at right angles to the plane of the paper; the moment of the force with respect to O is the measure of its tendency to make the body turn round this axis. For the sake of brevity, however, it is usual to speak of the force as tending to make the body turn round the point O. As a body can turn round a point in only two directions,[1] it is necessary to have some way of distinguishing these two directions; this is very conveniently done by reference to the hands of a watch supposed to be placed on the paper with its face upwards. It is plain that a body may turn round a point either in the same direction or in the opposite direction to the hands of the watch; thus, in fig. 31, the tendency of P is to turn the body on which it acts about the point O, in the direction contrary to that of the motion of the hands of a watch.

It is frequently convenient to indicate this difference of direction of rotation by a difference in the sign of the

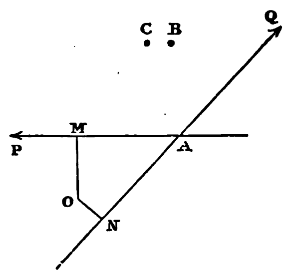
[1] i.e. When its motion is limited as above explained; otherwise it is capable of turning in an infinite number of ways round a point.

moment which measures the tendency, according to the following rule:—When a force tends to turn a body round a point in the same direction as that of the motion of the hands of a watch, its moment with regard to the point is reckoned negative; when in the contrary direction, its moment is reckoned positive.

43. *The principle of moments* is the name of a fact or theorem which may be thus stated:—If any forces act on a body in one plane, the sum of their moments with respect to any point in that plane equals the moment of their resultant with respect to the same point. It must be observed however that, in forming the sum, each moment must be written down with its proper sign, and then the sign of the sum will give the sign of the moment of the resultant, and will show the direction in which the resultant tends to turn the body round the point.

Ex. 21.—Let P and Q be forces of 8 and 10 units; and let a point O be taken such that the perpendiculars O M, O N, let fall on their directions, are 3 and 2 units of length respectively. These forces tend to turn the plane round O in the contrary direction to that of the motion of the hands of a watch; their moments (24 and 20) are therefore positive, and the moment of their resultant is 44, which, being positive, shows that their resultant also tends to turn the plane round O in the contrary direction to that of the motion of the hands of a watch.

FIG. 32.



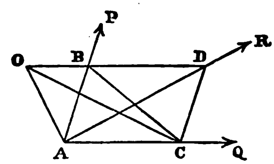
Ex. 22.—If in the last case we take a point within the angle P A Q, such that the perpendiculars on A P and A Q are 5 and 3·5 units of length respectively, the moments severally are -40 and $+35$; consequently the moment of the resultant about that point will be -5, and the resultant will tend to turn the body round that point in the same direction as that of the motion of the hands of a watch.

Ex. 23.—If in the last case a point is taken so that the former perpendicular is 5 but the latter 4·5 units, the moments severally are -40 and $+45$

consequently the moment of the resultant about that point is $+5$, and the resultant tends to turn the plane round this point in the contrary direction to that of the motion of the hands of a watch. The points referred to in Ex. 22 and 23 are indicated in the diagram by B and C.

44. *The proof of the principle of moments* in the case of two forces and their resultant can be easily derived from the principle of the parallelogram of forces; thus:—Let P and Q be two forces whose directions intersect in A; let O be the point with reference to which the moments are to be taken; draw O B parallel to A Q, and take A C such that

FIG. 33.



$$AC : AB :: Q : P;$$

then A B and A C will represent the forces P and Q; consequently if the parallelogram A B D C is completed, A D will represent the resultant (R) of P and Q. Join O A, O C and B C. The moments of P, Q, and R are proportional to the areas of the triangles O A B, O A C, and O A D (Art. 42). Now O A C is equal to B A C, which being half of the parallelogram, is equal to A B D. But O A D is made up of O A B and B A D. Consequently,

$$\text{moment of R} = \text{moment of P} + \text{moment of Q}.$$

In this case all the moments are positive; we will therefore take a case in which O is so situated that the moments of R and Q with regard to it are positive, and that of P negative, and in which consequently we have to show that

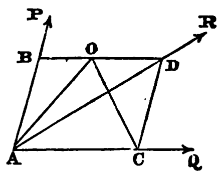
$$\text{moment of R} = -\text{moment of P} + \text{moment of Q}.$$

In this case draw O B parallel to A D, and find A C from the proportion

$$AC : AB :: Q : P;$$

so that A B and A C represent the forces P and Q, and on completing the parallelogram A B D C, A D will represent their resultant (R). Join O A, O C. We see that O A C is half the parallelogram, and consequently equals A D B, which is made up of O A D and O A B; hence

FIG. 34.



$$O\,A\,D = O\,A\,C - O\,A\,B;$$

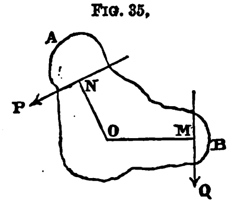
and as these triangles are proportional to the moments of the forces with respect to the point O, we have

$$\text{moment of } R = -\text{moment of } P + \text{moment of } Q.$$

Similar results are obtained for any other position of the point O; and the student will find it instructive to consider one or two other cases, e.g. that in which O falls within the angle R A Q, in which case the moments of P and R are both negative and that of Q positive. He will observe that, by the aid of the rule for the signs of the moments, all possible cases of two intersecting forces are included in the one statement given above. And we shall see in a subsequent article that the proposition is true in the case of two parallel forces and their resultant.

45. *Equilibrium of a body capable of turning round a fixed axis.*—Let P and Q be two forces acting on a body A B, capable of turning freely round an axis passing through O at right angles to the plane of the paper. O is a fixed point in the sense already explained (Art. 42). As P and Q keep the body at rest, their resultant must pass through O; consequently if moments are taken with respect to O, the moment of the resultant must be zero, and therefore the sum of the moments of P and Q

FIG. 35.



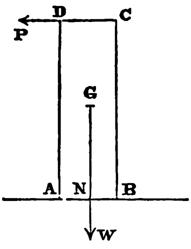
with respect to O must be zero; as the moments must have contrary signs, this condition is equivalent to the equation

$$P \times ON = Q \times OM,$$

O N and O M being lines drawn from O at right angles to the lines along which P and Q act. The same reasoning will apply to the case in which there are three or more forces, and thus we obtain the conclusion that the sum of the moments of the forces with respect to the fixed point is zero, each moment being written down with its proper sign. It is, however, frequently convenient to state the same result thus:—Take the sum of the moments of the forces which tend to turn the body from right to left round the point, and also the sum of the moments of the forces which tend to turn the body from left to right round the point; these two sums are equal. This rule is true whether the forces are or are not parallel.

Ex. 24.—A B C D is a rectangular block of sandstone 10 ft. high, with a base 3 ft. square; what force, P, acting along C D will overthrow it?

FIG. 36.



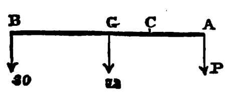
The forces are P and the weight of the block (W) acting vertically downward through its centre of gravity G. If P is just not sufficient to overthrow the body, the forces will be in equilibrium on the point A; and we see that they tend to turn the body in opposite directions round A. Consequently

$$P \,.\, AD = W \,.\, AN,$$

or, as AD is 10 ft., AN 1·5 ft., and W 14,062·5 lbs. (Art. 5), we find that P must be a force of 2,109$\frac{3}{8}$ lbs.

Ex. 25.—A B is a rod of uniform section weighing 12 lbs. and 10 ft. long; it can turn freely about a point (C) 3 ft. from A; a weight of 30 lbs. is hung at B; what weight must be fastened to A that the rod may be in equilibrium?

FIG. 37.



Let P be the required weight, its moment is 3 P; the weight of the rod may be supposed to act vertically through the centre of gravity G (Art. 20), and its

moment is 2×12; the moment of the weight at B is 7×30. Now P tends to turn the rod in one direction round C, while the forces 30 and 12 tend to turn it in the opposite direction; consequently

$$3P = 2 \times 12 + 7 \times 30,$$

or $$P = 78 \text{ lbs.}$$

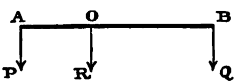
If the proper signs were prefixed to the moments they would be $-3P + 2 \times 12$ and $+7 \times 30$; and then the equation would be

$$-3P + 2 \times 12 + 7 \times 30 = 0,$$

which is plainly the same as the equation used above. The student would find it a useful exercise to solve Ex. 19 in the same manner.

46. *Resultant of two parallel forces.*—When two forces act along parallel lines they are called parallel forces; their resultant is found by the following rules:—(*a*) Let P and Q be any two parallel forces acting in the same direction;[1] draw the line AB at right angles to AP and BQ, and in it take a point O, such that

FIG. 38.

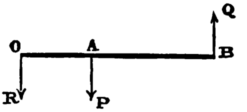


$$OA \cdot P = OB \cdot Q;$$

draw OR parallel to AP and BQ; the required resultant (R) is a force equal to $P+Q$ acting along OR in the same direction as P and Q.

(*b*) Let P and Q act in opposite directions; and let P be the greater; in BA produced take a point O, such that

FIG. 39.



$$OA \cdot P = OB \cdot Q;$$

draw OR parallel to AP and BQ; the required resultant equals $P-Q$ and acts along OR in the same direction as P.

[1] The student should notice this use of the word direction (see Art. 8). If two forces act along parallel lines from left to right, they act in the same direction; but if one acts from left to right and the other from right to left, they act in opposite or contrary directions.

Ex. 26.—Suppose that in case (a) P and Q are forces of 10 and 8 units, and that A B is 45 in. long; the resultant will be a force of 18 units, and the point O will be at a distance of 20 in. from A, and therefore of 25 in. from B.

Ex. 27.—Everything else being as in the last example, suppose the direction of Q to be reversed so that the forces act as in case (b); the resultant will now be a force of 2 units, acting in the same direction as P through a point O, distant 180 in. to the left of A.

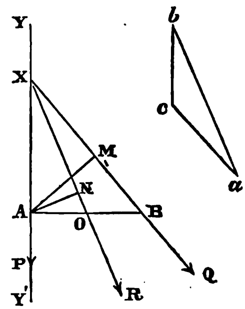
Ex. 28.—In the last case suppose Q to be a force of 9·9 units, other conditions continuing the same; the resultant will be a force of $\frac{1}{10}$th of a unit, acting at a distance of 4455 in. to the left of O. The student will observe that when P and Q are very nearly equal, the resultant is a very small force acting along a very distant line parallel to the lines along which the forces act.

47. The rules stated in the last article can be easily shown to follow from the parallelogram of forces. It must be premised, however, that it is open to us to reason as follows:—Suppose P A Y (fig. 40) to be a line fixed in position, and suppose a second line Q B drawn through a fixed point B to be produced to cut the former line in X; suppose X to be moved further and further from A towards Y, making the line X B Q turn round B; as X gets more and more remote from A, the line X B Q will continually approach a state of parallelism with A Y; and if we suppose X to become indefinitely remote from A, the lines will become actually parallel. Bearing this in mind, let P and Q be two forces acting along lines A P, B Q which intersect in X, and let their resultant (R) act along the line X R; also suppose B to be so chosen that A B is at right angles to A P; draw lines A N and A M at right angles to X R and X Q. Now if moments are taken with respect to the point A, the moment of P is zero; consequently we know that A M. Q equals A N. R (Art. 44).

Again, draw the triangle $a\,b\,c$, whose sides $b\,c$, $c\,a$, are parallel to A P and B Q, and proportional to the forces P and Q; then we know that $a\,b$ is proportional to the force that would balance P and Q; and therefore also to their resul-

tant (Art. 40). It follows, therefore, that the sides of the triangle $a\,b\,c$ are proportional to the three forces P, Q, R. So far we have stated nothing but what has been laid down in preceding articles. Now all this is true irrespectively of the position of X; suppose then that X is moved further and further from A along A Y towards Y, the figure will undergo no essential change, but B Q will become more and more nearly parallel to A P, and A M and A N will more and more nearly approach coincidence with A B, while in the triangle $a\,b\,c$, the angle $a\,c\,b$ will continually approximate to two right angles and $a\,b$ become more and more nearly equal to $b\,c + c\,a$. When X becomes indefinitely distant from A, P and Q will become parallel forces acting towards the same part, and we shall now have (1) $b\,c+c\,a=a\,b$, i.e. P+Q = R; and if we suppose the point N to come ultimately to O, we shall have (2) A O . R=A B . Q, i. e. A O (P+Q)=A B . Q or A O . P =O B . Q. These results coincide with those stated in first rule (a) of the preceding article. If we suppose the point X, instead of moving towards Y, to move in the contrary direction towards Y′, the results obtained will coincide with those stated in the second rule (b) of the preceding article; and consequently these rules are proved to be true.

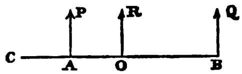
FIG. 40.



48. It follows immediately from the last article that the sum of the moments of any two parallel forces with respect to any point in their plane equals the moment of their resultant with reference to the same point. Let P and Q be two forces acting along parallel lines, as shown in the figure; take any point C in the plane of the paper;

draw the line C A B at right angles to A P and B Q. If R is the resultant of P and Q, we know that R equals P + Q and acts along a line O R at right angles to A B, the point O being so chosen that P . A O equals Q . O B. Now

FIG. 41.



$$\text{CO} \,.\, \text{R} = \text{CO} \,.\, \text{P} + \text{CO} \,.\, \text{Q}$$

$$= (\text{CA} + \text{AO})\,\text{P} + (\text{CB} - \text{OB})\,\text{Q};$$

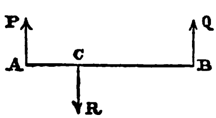
and since $\text{AO} \,.\, \text{P} - \text{OB} \,.\, \text{Q} = 0,$

we have $\text{CO} \,.\, \text{R} = \text{CA} \,.\, \text{P} + \text{CB} \,.\, \text{Q}.$

But C O . R, C A . P, and C B . Q are the moments with respect to C of R, P and Q respectively, so that the above equation shows that the moment of R equals the sum of the moments of P and Q. There are several cases, but all can be included in one common enunciation, by a method similar to that employed in the above case. The student should work out some of these cases, e.g. (1) Suppose C to be between A and O, the point to be proved will be that C O . R equals − C A . P + C B . Q. (2) Suppose P to be greater than Q, and Q's direction to be reversed, R will equal P − Q, and O will fall to the left of A; if we suppose further that O is between C and A, the point to be proved will be that C O . R equals C A . P − C B . Q.

49. *Condition of equilibrium of three parallel forces.* —The three forces will be in equilibrium if any one of them is equal and opposite to the resultant of the other two. Accordingly, let P, Q and R be the three forces, of which R is the greatest, and let a line be drawn at right angles to the lines along which they act, cutting them in A, B, C, respectively; then, that the forces may be in equilibrium, the following conditions must be fulfilled: (1) P and Q must act in the same direction, and R

FIG. 42.

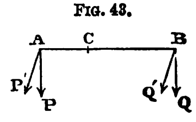


in the opposite direction; (2) R must equal P+Q. (3) The moment P . C A must equal Q . C B; it is also necessary that these moments have opposite signs, and this is equivalent to a fourth condition: (4) the point C must fall between A and B. It is to be observed that, instead of the third condition, we may have either R . A C equal to Q . A B, or R . B C equal to P . B A; but these are not independent conditions, for any two of them may be deduced from the third; and it is an instructive exercise to deduce them. Another deduction of some importance may also be left as an exercise for the student:—In Articles 46–49 it has been assumed that the line A B is drawn at right angles to the lines along which the forces act; if, however, A B were inclined at any angle to those lines, we should still have P . C A equal to Q . C B. Of course in this case P . C A is not the moment of P with reference to C; it is, however, a quantity proportional to the moment.

Ex. 29.—If P and Q are forces of 5 and 7 units acting at A and B along parallel lines in the same direction, and balanced by a force R acting through C, then (1) R must act in the opposite direction to P and Q, (2) it must be a force of 12 units, (3) C must be between A and B, and (4) C A must be $\frac{7}{12}$ths of A B. All these points must hold whether the line A B is at right angles or not to P A, Q B, and R C.

50. *Centre of two parallel forces.*—Let P and Q be the forces acting at A and B; join A B, and let C be the point in A B through which their resultant acts. Now suppose that A P and B Q are turned round A and B through any equal angles into the positions A P′ and B Q′, so that the forces continue parallel and otherwise unchanged; their resultant will still pass through C.

FIG. 43.



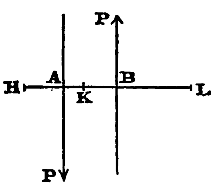
This point, which remains fixed relatively to A and B, in spite of the change in the direction of the parallel lines, is called the centre of the two parallel forces. If there are three or more parallel

forces acting on points relatively fixed, they will have a centre, i. e. there will be a point fixed relatively to the other points, through which the resultant will act in spite of a change in the direction of the parallel forces.

If P and Q were the weight of two masses placed at A and B, C would be the centre of gravity of the masses; the centre of gravity of the masses, therefore, coincides with the centre of the two parallel forces, viz. their weights; and the like is true of three or of any number of points.

51. *Statical couples.*—Two equal forces acting along parallel lines in opposite directions make a combination which is called a couple, or a statical couple. A couple has the property that, if the moments of the forces composing it are taken with respect to any point in their plane, the sum of the moments is constant both in sign and magnitude; thus, let the couple consist of the two equal forces P as shown in the figure; draw any line H K L at right angles to their direction, and cutting the lines of action of the forces in A and B; the sum of the moments of the forces with respect to any point in the plane is A B . P, and in the case shown in the figure is positive. This is evidently true since the sums of the moments of the forces about H, K and L are respectively

FIG. 44.



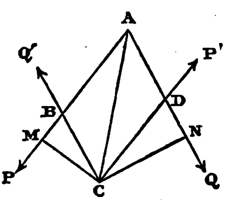
$$-\text{HA} \cdot \text{P} + \text{HB} \cdot \text{P},\ \text{KA} \cdot \text{P} + \text{KB} \cdot \text{P}, \text{ and } \text{LA} \cdot \text{P} - \text{LB} \cdot \text{P};$$

and each of these is clearly equal to A B . P. If the forces had their directions reversed without undergoing any other change, the sum of their moments with respect to any point in their plane would be $-$A B . P. The line A B is called the arm of the couple, and the product A B . P with its proper sign is called the moment of the couple.

A couple has the property that the forces composing it cannot be replaced by a single resultant force; this can be deduced from Art. 46 (*b*), from which it can be easily shown that if P is larger than Q by an exceedingly small quantity, their resultant would be an exceedingly small force acting along a parallel line exceedingly distant from P and Q (compare Ex. 28); consequently when the forces become actually equal, the resultant becomes zero and acts along an infinitely distant line; in other words, no such resultant exists. It will be observed that a couple is the only case of two forces acting in the same plane which cannot be replaced by a single resultant.

52. *Equilibrium of two couples.*—If two couples acting in the same plane have equal moments of opposite signs, they will be in equilibrium; this can be easily proved as follows:—Let P and P′ be the forces forming the one couple, and Q and Q′ the forces forming the other; the lines along which they act will, by their intersection, form a parallelogram A B C D; and as the moments of the couples have opposite signs, the forces must act as shown in the figure.

FIG. 45.

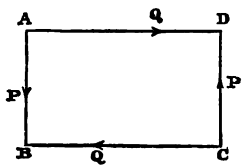


Draw C M and C N at right angles to A P and A Q, then P . C M is the moment of the one couple and Q . C N is the moment of the other, and these moments are equal. Now if P is represented by A B, the area of the parallelogram represents the moment of P, and therefore it equally represents the moment of Q; but as the area equals A D . C N, we see that the force Q is represented by A D. Now, as A B and A D represent the forces P and Q, A C will represent their resultant. In the same way, by drawing perpendiculars from A on C P′ and C Q′, it may be shown that the resultant of P′ and Q′ is represented by

C A. Hence, the four forces are equivalent to two equal forces acting in opposite directions along the line A C; and consequently these four forces, which make up the two couples, must be in equilibrium. This result is true irrespectively of the magnitude of the parallelogram, or of that of its angles; the case, therefore, in which the four forces act along parallel lines is included as an extreme case in the above proof, and the statement at the head of the article is true in all cases.

Ex. 30.—A B C D is a rectangle whose sides A B and B C are 3 in. and 5 in. long respectively, equal forces (P) of 15 units act from A to B and C to D, and equal forces (Q) of 25 units act from A to D and C to B. Why must these four forces be in equilibrium?

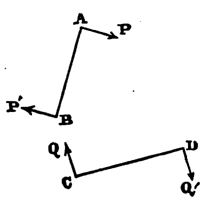
FIG. 46.



We see that the forces P make up a couple with a moment + 75, while the forces Q make up a couple with a moment − 75; as these couples have equal moments of opposite signs, they (i.e. the four forces) must be in equilibrium.

53. *Equivalence of two couples.*—If there are any two couples whose moments are equal and of the same sign, acting on a rigid body in one plane, either would be balanced by a couple of equal moment and of opposite sign; it follows therefore that any two couples of equal moments of the same sign and acting in the same plane are equivalent to one another, however else they differ. The same fact may be stated thus:—A couple may be transferred to any position in its plane, and its arm may be lengthened or shortened in any way, provided its moment and the sign of it remain unchanged, e.g. if P and P′ are two forces of 12 units forming a couple whose arm A B is 5 units (say

FIG. 47.

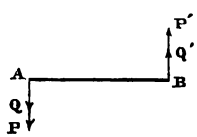


nches) long; and Q and Q′ are two forces of 10 units orming a couple whose arm C D is 6 in. long; these ouples have equal moments of the same sign (viz. -60) nd consequently the one could be used to replace he other, whatever positions they may occupy in the lane.

Ex. 31.—Let A B C D be a rectangle of the same dimensions as in Ex. 30, nd let it be acted on by two forces of 12 units, the one from A to B, the ther from C to D. We might take away these forces, and use instead of em two forces of 20 units, one acting from B to C, the other from D to A. 'his is plain, since each pair of forces is a couple, the moment of the former eing $+12 \times 5$ and of the latter $+20 \times 3$, i.e. $+60$ in both cases.

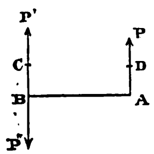
54. *The resultant of two couples* is a couple whose oment is the sum of the moments of the two couples, heir signs being taken into account; thus, if the moments f the separate couples are $+60$ and $+50$, pounds and et being used as units, we may take an arm (A B) 10 feet ong, and suppose forces P and P′ each of 6 lbs. to act at and B, as shown in the figure; they are equivalent to the ormer couple, since their moment s $+60$. If we suppose two forces Q nd Q′ each of 5 lbs. to act at A and , they are equivalent to the latter ouple, since their moment is $+50$; he two together are equivalent to wo forces of 11 lbs. each, acting at and B, i.e. to a couple whose moment is 110 or $60+50$. f the moments of the couple had been -60 and $+50$, we ust suppose the directions of P and P′ reversed, and the ur forces will be equivalent to two forces of one pound piece acting at A and B, forming a couple whose moment -10, which is equal to $-60+50$, i.e. the sum of the vo moments taken with their proper signs. As the same easoning plainly applies to any other case, the statement t the head of the article is manifestly true.

FIG. 48.



55. *Transfer of a force along a parallel line.*—Let P be any force acting along the line A D; draw any line B C parallel to A D, and let A B be at right angles to both. Suppose forces P′ and P″ equal to P to act in opposite directions along B C; these forces will be in equilibrium. Consequently, the three equal forces are equivalent to P. Now as P and P″ form a couple whose moment is A B . P, it follows that P acting along A D is equivalent to an equal force acting in the same direction along B C, together with a couple whose moment is A B . P. The couple in this case has a positive moment; if the line B C had been on the other side of A D, i.e. to the right of a person looking in the direction of P's action, the couple introduced by the transfer of P would have had a negative moment.

FIG. 49.

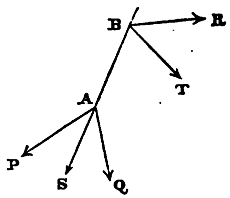


On the other hand, if a force and a couple act on a rigid body in the same plane, they are equivalent to a force equal to and acting in the same direction as the given force; but along a parallel line whose position is determined by the magnitude and sign of the moment of the couple. Thus, suppose the units are pounds and feet, and that a body is acted on by a force (P′) of 8 lbs. and a couple whose moments is + 40. The couple will be equivalent to two forces (P″ and P) of 8 lbs. each, acting in opposite directions along parallel lines 5 feet apart; the couple may be placed in such a position that one of the forces (P″) may act along the same line, but in an opposite direction to P′, as shown in the figure, the line A B being supposed to be 5 feet long. When this has been done, it is plain that the force P′ and the couple are together equivalent to the single force P, which is a force of 8 lbs. and acts as above stated.

56. *Resultant of three or more forces.*—When three or

more forces act in one plane on a rigid body, they may be in equilibrium; but if not, they can be replaced either by a single resultant or by a couple. Thus, if P, Q, R are three forces acting in one plane on a rigid body, we may produce the lines along which P and Q act to meet in A, and by Art. 37 find the resultant S of P and Q; the three forces are now replaced by the two S and R; we can produce the lines of their action to meet in B, and then, by Art. 37, find the resultant T of S and R; this force is the required resultant of P, Q and R. This process may be varied in several ways, e.g. we may begin by finding the resultant of P and R, and then find the resultant of the force so found and Q. It may happen that P and Q are parallel forces; in that case the force S must be found by Art. 46. If S and R are found to be equal parallel forces acting in opposite directions, they will form a couple; under all other circumstances P, Q and R acting in one plane can be replaced by a single resultant or will be in equilibrium.

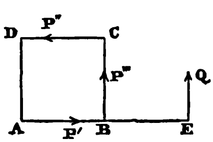
FIG. 50.



Ex. 32.—Let ABCD be a square, and let three equal forces, P' P''' P'', each of 20 units, act from A to B, from B to C, and from C to D; find their resultant.

Since P' and P'' are a couple, they can be placed in such a way that one of them shall act along CB in an opposite direction to P'''; when this is done the other will act as indicated by Q, through a point E so taken that BE equals AB. This force, which will equal 20 units, is the resultant required. The student should make this out, and then consider the following results:—(*a*) Let $P'=P''=14$ units, and $P'''=7$ units, the resultant will be a force of 7 units acting like Q, but BE will be the double of AB; (*b*) let $P'=P''=P'''=14$ units, but let P''' act in the opposite direction, viz. from C to B; the three forces are equivalent to a force of 14 units acting from D to A; (*c*) in case *b*, if P''' were a force of 28 units, the three forces

FIG. 51.



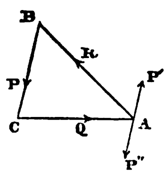
would be equivalent to one of 28 lbs. acting in the direction D to A along the line which bisects D C and A B.

Ex. 33.—Let A B C D be a square, and let forces of 6 units act from A to B, C to D, and A to D respectively, and let one of 8 units act from C to B; what is the resultant of the four forces?

Since the forces acting from A to B and C to D are a couple, they can be transferred so as to act along D A and B C respectively; we now have these forces:—6 units from A to D, 6 units from D to A, 6 units from B to C, and 8 units from C to B, which are clearly equivalent to a force of 2 units acting from C to B.

Ex. 34.—Let A B C be a triangle whose sides, B C, C A, A B, are respectively 5, 6, and 7 in. long. Let forces P, Q, R, of 15, 18, and 21 units, act from B to C, C to A, and A to B, as shown in the figure. Find their resultant.

FIG. 52.



The forces are proportional to the sides of the triangle; this suggests the following solution:—Suppose two forces (P' P'') of 15 units apiece to act at A in opposite directions along a line parallel into B C; as these are in equilibrium, the five, P, P', P'', Q, R, are equivalent to the original three forces; but by Art. 40, P'', Q, R are in equilibrium; consequently the original three forces are equivalent to two forces (P and P') of 15 units each forming a couple. As the perpendicular let fall from A on B C is 5·879 in., the moment of the couple is 88·2.

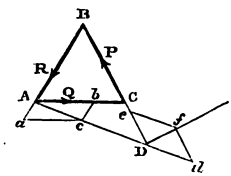
If the sides of the triangle in this example are of any length, and each force proportional to the side along which it acts, a similar result will be arrived at, viz. that the three forces are equivalent to a couple whose moment is proportional to twice the area of the triangle. The student will easily prove this, and he should note carefully the difference between this case—in which the three forces act along the sides of the triangle—and that discussed in Art. 40, where the forces act on a point along lines parallel to the sides of a triangle.

57. *Remark.*—When three or more forces are given it will commonly happen that several means may be adopted for finding their resultant, e. g. any of the Examples from 32 to 34 might be treated by the method suggested in Art. 56, and the student will find it an instructive exercise to work out some of these examples by that method. We *here* add one example which differs from those referred to

in this respect, that its solution does not admit of obvious simplification by the properties of couples.

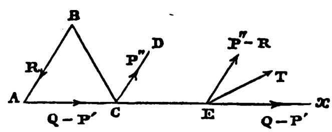
Ex. 35.—Draw an equilateral triangle A B C, and let forces P, Q, R, of 8, 12, and 4 units act along its sides from C to B, A to C, and B to A respectively; find their resultant.

FIG. 53.



In B A produced take A *a* of any length, and from A C cut off A *b* equal to 3 A *a*; these lines represent the forces R and Q, and their resultant is represented by A *c*; produce A *c* and B C to meet in D, make D *d* equal to A *c*, and D *e* to 2 A *a*; we may suppose P and the resultant of Q and R to act at D, consequently P and that resultant are represented by D *e* and D *d*, and by completing the parallelogram D *e f d*, the required resultant of P, Q and R, will be represented by D *f*, and will be found to be a force of 6·93 units. The resultant can also be found thus:—Draw C D parallel to A B, and resolve P into two components, P' and P'', along C A and C D; these will be forces of 8 units apiece. We now have a force $Q-P'$ of 4 units acting from A to C, a force R of 4 units from B to A, and a force P'' of 8 units from C to D. As R and P'' are parallel forces acting in opposite directions, we can (by Art. 46) find their resultant to be a force $P''-R$ of 4 units, acting through E as shown in the figure, E being taken so that C E equals A C. The forces are now reduced to two forces ($P''-R$ and $Q-P'$) of 4 units each, acting as shown in the figure. Their resultant T can now be found, and the result is that T E *x* is an angle of 30°, and T a force of 6·93 units. The student should carefully work out both cases to scale, and ascertain that the same result is obtained by either method.

FIG. 54.



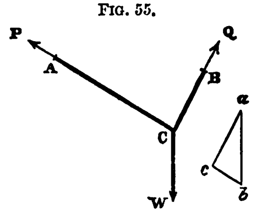
58. We will now apply the principles already explained to the solution of a number of questions of practical interest, which may be comprised under the following heads:—(*a*) Tensions of threads. (*b*) Thrusts and tensions of rods. (*c*) Equilibrium of levers, including the balance. (*d*) Equilibrium of bodies resting on planes. (*e*) Tendency of forces to bend and break rods, and some other applications of couples. We shall discuss each par-

ticular question briefly, and in many cases merely indicate the method by which the student may obtain for himself the desired solution.

59. *Tensions of threads.*—It will be assumed that the threads are perfectly flexible and without weight.

Ex. 36.—A weight W is fastened by a knot to a point C of a thread which hangs from two points, A and B; find the tensions of the thread.

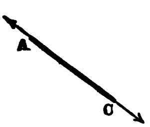
FIG. 55.



As A and B are known points, and A C and B C of known lengths, the positions of A C and C B are known; also C W is vertical. If P and Q are the reactions of the fixed points A and B, they will be transmitted to C, and the point C will be kept at rest by the forces P, Q, W, as shown in the figure. Draw $a\,b$, $b\,c$, $c\,a$ parallel to C W, C A, and C B; if $a\,b$ contains as many units of length as W contains units of force (say lbs.) $b\,c$ and $c\,a$ contain as many units of length as P and Q contain pounds. These are therefore known, and are the tensions of A C and B C.

Ex. 37.—In the above case let A B be horizontal and 10 ft. long, A C and C B 8 ft. and 6 ft. long respectively, and W a weight of 100 lbs. It will be found that P and Q are forces of 60 lbs. and 80 lbs. respectively.

FIG. 56.



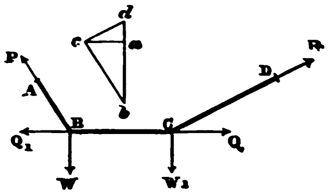
The student will observe that the thread A C is stretched by two equal forces of 60 lbs., one the reaction of the fixed point A acting at A, the other at C due to the action of W; these forces act as shown in fig. 56. It is the former force transmitted through the thread to C which is shown by P in fig. 55. In like manner B C is stretched by two equal forces of 80 lbs.

Ex. 38.—A thread is suspended from points A and D; weights W and W_1 are hung from points B and C of it; the positions of A B, B C, C D are known, and also the weight W; it is required to find W_1, and the tensions of the parts of the thread.

Let P and R be the reactions of the points A and D; the former is transmitted through the thread to B; so that the point B is acted on by two forces, P and W, in the directions shown in the figure. Similarly the point C is acted on by the forces R and W_1 in the directions shown in the figure. Now the tendency of P and W is to make B move to the left, while that of R and W_1 is to make C move to the right. These opposite tendencies exactly neutralise each other by transmission along the thread B C. Let Q be the

force transmitted from C in consequence of the action of R and W_1; then the point B is kept at rest by the three forces P, Q, W. In like manner, if Q_1 is the force transmitted from B in consequence of the action of P and W, the point C is kept at rest by the forces R, W_1, Q_1. As the thread BC is at rest, we must have Q equal to Q_1. Draw a vertical line dab; let ab contain as many units of length as W contains pounds; draw bc, ca parallel to BA, BC; these lines give P and Q; consequently ac also gives Q_1. Draw cd parallel to CD, then cd and da will give R and W_1.

FIG. 57.

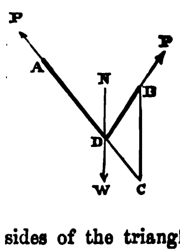


Ex. 39.—If BC is horizontal, and AB and DC inclined to the horizon at angles of 57° and 25° respectively, and if W is a weight of 100 lbs., it will be found that P, Q, R, and W_1 are forces of 119·3, 65, 71·7, and 30·3 lbs. The student will observe that in this case W and P act on BC at B in the direction C to B with a force of 65 lbs., while W_1 and R act on BC at C in the direction B to C also with a force of 65 lbs.; BC is therefore stretched by these two forces. Similarly AB is stretched by two equal forces of 119·3 lbs., and CD by two of 71·7 lbs. So that the tensions of AB, BC, CD are 119·3 lbs., 65 lbs., and 71·7 lbs, respectively.

Ex. 40.—A thread of given length (a) is fastened to two given points; it passes through a smooth ring carrying a weight (W); find the position in which it will come to rest, and the tension of the thread.

The tension of the thread must be the same throughout, so that the reactions at A and B must be equal forces, which, being transmitted through the thread, support the weight W. The thread must therefore come into such a position that the angle between its two parts shall be bisected by a vertical line.

FIG. 58.



Let A and B be the two points; draw a vertical line BC; with centre A and radius a describe a circle cutting BC in C; make the angle CBD equal to ACB; then DB will equal DC (Eucl. i. 6), and ADB is the position of the thread. For if a vertical line NDW be drawn through D, it is plain that the angles ADN and BDN are equal. If P is the tension of the thread, D is in equilibrium under the action of the three forces W, P, and P. Also it is plain that these forces are proportional to the sides of the triangle BCD, i.e.

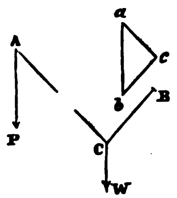
$$P : W :: DB : BC.$$

Ex. 41.—Let AB be horizontal and 12 ft. long, the length of the thread

20 ft., and W a weight of 150 lbs.; it will be found that the tension of the thread is a force of 93¾ lbs.

Ex. 42.—A thread is fastened to the point B, and has a weight P (of 10 lbs.) fastened to the other end; it passes over a smooth point A, and through a smooth ring carrying a weight W (of 15 lbs.); find the position in which the whole will come to rest.

FIG. 59.

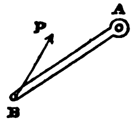


Here a weight of 15 lbs. is supported by two forces of 10 lbs. Draw any vertical line $a\,b$, containing 15 units of length, and construct the triangle $a\,b\,c$, whose other sides are 10 units apiece; three forces of 15, 10, and 10 lbs. apiece acting on a point parallel to the sides of this triangle will keep the point at rest; draw B C and A C parallel to $b\,c$ and $a\,c$, and draw the vertical lines A P and C W; then P A C B is the position of the thread, and C that of the ring. The forces acting on the ring at C are (1) the force transmitted to C from P (10 lbs.) in the direction C A, (2) the force transmitted to C from the reaction of B (10 lbs.) in the direction C B, (3) the weight W (15 lbs.)

Ex. 43.—Suppose the line A B to be inclined at an angle of 30° to the horizon, and suppose P and W to be 50 and 80 lbs. respectively; find (1) the position in which W will come to rest, (2) the magnitude and direction of the pressure on A; and show that if A and B are 12 ft. apart the thread must be more than 17·3 ft. long.

60. *Thrusts and tensions of rods.*—In questions regarding a framework of rods we make the following suppositions:—(1) that the rods are without weight; (2) that they are without thickness; (3) that the joint connecting any two rods is made by means of a perfectly smooth rivet. The consequence of these suppositions is that the force transmitted from joint to joint passes wholly along the rod; e.g. suppose A B to be a rod capable of turning quite freely round the point A, suppose it without weight, and suppose it to be acted on at the end B by a force P. The only forces keeping the rod at rest are P and the reaction of A; these must act along A B. If the force P were to act as shown in the figure, it would make the rod turn

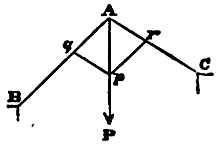
FIG. 60.



round A. If the rod were heavy, the resultant of P and the weight of the rod would have to pass through A. If the rivet were exceedingly rough, the force P might act along a line passing considerably to the right or left of A without causing the rod to turn.

Ex. 44.—Let A B and A C be two rods loosely jointed at A, and let their ends rest at B and C; a weight P is hung at A; find the pressures produced at B and C.

FIG. 61.

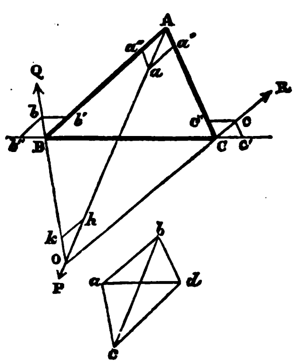


Take A p to represent P; complete the parallelogram A $q p r$; A q and A r represent the components of P along A B and A C; these can be transmitted only in the direction A to B and A to C; consequently they are the pressures exerted at B and C.

Ex. 45.—If A B and A C are inclined to the vertical at angles of 45° and 60° respectively, and if P is a weight of 500 lbs., there will be a pressure of 448·3 lbs. exerted at B along A B, and a pressure of 366 lbs. at C along A C.

Ex. 46.—A triangular frame A B C consisting of rods loosely jointed at the angles is in equilibrium under the action of three forces acting one at each angle; it is required to find the thrust or tension to which each rod is subjected.

FIG. 62.



Let the forces P, Q, R act at the angles A, B, C respectively, and suppose their directions to pass through a point O. In the first place they must be in equilibrium, or otherwise they would make the frame itself move; draw $h\,k$ parallel to O R, then we know that the forces P, Q, R are proportional to h O, O k, and $k\,h$. Take A a, B b, C c equal to O h, O k, and $k\,h$ respectively, and complete the parallelograms A $a'\,a\,a''$, B $b'\,b\,b''$, C $c'\,c\,c''$; it will be found that A a'' = B b', B b'' = C c', and C c'' = A a'. We see therefore that each rod is under the action of a pair of equal forces which tend to crush A B and A C, and to stretch B C; these forces are severally proportional to A a'', C c'', and B b''. These lines therefore measure the thrusts of A B and A C and the tension of B C. The magnitudes of these forces can be obtained by a more simple construction, thus:—Draw $b\,c$ parallel to O A, and containing as many units of length as P contains units of force; draw

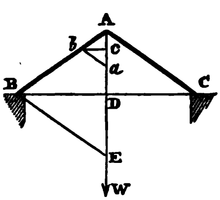
ca, ab parallel to OB and OC respectively; draw ad parallel to BC, dc to AB, and join bd (the line bd is parallel to AC, as the student can prove), and we shall have ca, ab, ad, dc, bd containing as many units of length respectively as the forces Q, R, the tension of BC, and the thrusts of AB and CA contain units of force. This is plain from an inspection of the figure. Since abc is the triangle of forces for the three forces in equilibrium at the point O, cda for the three forces at the point B, dab for the forces at C, and bcd for those at A.

Ex. 47.—Draw an equilateral ABC; draw the lines OB, OC, making with BC angles OBC and OCB of 30° and 40° respectively, O and A being on opposite sides of BC; suppose ABC to be a framework of three equal rods, and that a force of 1,000 lbs. acts on the angle A along the line AO, balanced by forces Q and R acting on the angles B and C along the lines OB, OC; it will be found that Q is a force of 865 lbs., R of 878·2 lbs., and that the tension of BC equals 998·8 lbs., the thrusts of AB and AC 499·4 lbs. and 651·9 lbs. respectively.

The student will particularly observe that in this case the line BC is stretched by two forces of 998·8 lbs. apiece, one produced by Q and the other by R; the line AB is compressed by two forces of 499·4 lbs. apiece, one produced by P and the other by Q; and the line AC is compressed by two forces of 651·9 lbs. apiece, one produced by P and the other by R.

Ex. 48.—Let ABC be an isosceles framework sustaining a weight W at the point A, and resting on two points BC in the same horizontal line; find the pressures on the points B and C, and the forces transmitted along the rods.

FIG. 63.



Take Aa to represent W; draw ab parallel to AC; draw bc parallel to BC; then Ab will give the thrust of AB, bc the tension of BC, and Ac the pressure on the point of support B. It is plain that the thrust of AC equals that of AB, and the pressure on C equals that on B; and since Ac=ca, the pressure on B equals $\frac{1}{2}$ W. This is evident, since Aab is the triangle of forces for the three forces in equilibrium at A, and Abc the triangle for the three forces at B.

If a line is drawn through B parallel to AC, cutting AW in E, it will be evident that the weight W, the tension of BC, and the thrust of either AB or AC, are proportional to the lines AE, BD, and AB. The figure represents a roof, the feet of the rafters of which are tied by the tie-beam BC; we see therefore that the weight at the summit of the roof, the thrust of the rafter, and the tension of the tie-beam are respectively proportional to 2 AD (twice the pitch), AB (the length of the rafter), and $\frac{1}{2}$ BC ($\frac{1}{2}$ span).

Ex. 49.—If AB and AC are each 20 ft. long, and BC 32 ft. long, and the *weight at* A *half a ton*, we find that the thrust of the rafters equals 933 lbs.,

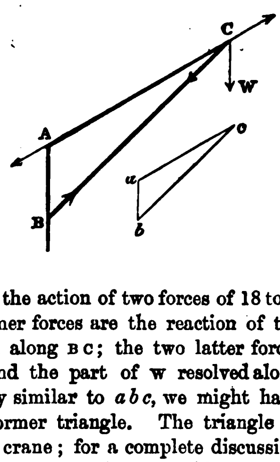
and the tension of the tie-beam 747 lbs., while the weight supported by each wall is 560 lbs.

If the tie-beam were cut there would (in this case) be at B and C a horizontal outward pressure of 747 lbs. exerted on each wall and tending to overthrow it; the use of the tie-beam is to enable these two forces to neutralise each other; so that the wall may be relieved from every pressure except the vertical pressure due to the weight of the roof.

Ex. 50.—A B is a block firmly fixed in the ground, to which two rods A C and B C are attached; A B, B C, and C A are 8 ft., 30 ft., and 24 ft. long respectively; a weight of 6 tons is hung at C; required the forces transmitted along the rods.

As the forces which support W must be transmitted along the rods, we may proceed thus:—Draw $a\,b$ parallel to C W and containing 6 units of length, draw $a\,c$ and $b\,c$ parallel to A C and B C; these lines will contain 18 and $22\frac{1}{2}$ units of length respectively, and consequently 18 and $22\frac{1}{2}$ tons respectively are the forces transmitted along the rods. The directions of the forces which support W are indicated by the arrow-heads near A and B; the forces which W exerts on the rods are represented by the arrow-heads near C; we see therefore that B C is under the action of two forces of $22\frac{1}{2}$ tons each tending to crush it, and A C is under the action of two forces of 18 tons each tending to stretch it. The two former forces are the reaction of the fixed point B and the part of W resolved along B C; the two latter forces are the reaction of the fixed point A and the part of W resolved along A C. Since the triangle A B C is evidently similar to $a\,b\,c$, we might have determined the forces at once from the former triangle. The triangle in this case is of the kind which occurs in a crane; for a complete discussion of the equilibrium of a loaded crane several other points would have to be taken into account which our limits will not allow us to enter upon. It may be mentioned, however, that two tension rods (such as A C) are used to ensure stability in other vertical planes besides that corresponding to the plane of the paper.

FIG. 64.

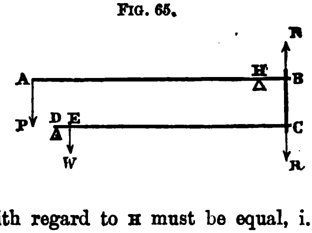


61. *Levers.*—A lever, considered as a statical implement, is merely a rigid rod kept in equilibrium by the action of three or more forces; in most cases it rests on a fixed point or fulcrum round which it can turn, the reaction of this point being one of the forces. If the

relation between the forces, other than this reaction, is all that is required, it is at once given by taking their moments with respect to the fulcrum, as in Ex. 25. A lever is commonly used to raise a weight or overcome some other resistance; accordingly, the two forces which act on the rod in addition to the reaction of the fulcrum are commonly called the power and the weight (see Art. 73). If the fulcrum is between the power and the weight, the lever is of the first order; if the weight is between the power and the fulcrum, the lever is of the second order; if the power is between the weight and the fulcrum, the lever is of the third order.

Ex. 51.—A B and C D are two levers, turning round fulcrums H and D; the ends B and C are connected by a link B C; a weight (W) of 3000 lbs. at E is supported by P acting at A; find P, having given the following dimensions—A H 60 in., B H 6 in., D E 4 in., D C 60 in.

FIG. 65.

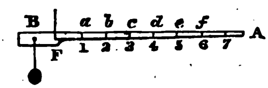


W tends to make the end C fall, it therefore exerts a force (R) at C which is transmitted along the link to B, and is balanced by P; consequently the moments of R and P with regard to H must be equal, i.e. $60 P = 6 R$ or $R = 10 P$. In like manner P tends to make B rise; it therefore exerts a force R′ at B which is transmitted along the link to C and is balanced by W; consequently the moments of R′ and W with regard to D must be equal, i.e. $60 R' = 4 W$ or $R' = \frac{1}{15} W$. Now as the link is at rest the forces R and R′ must be equal, and therefore $10 P = \frac{1}{15} W$ or $P = 20$ lbs., and R or R′ equals 200. The link is stretched by two forces of 200 lbs. apiece.

In the case of the lever A B the force P would be reckoned as the power and R as the weight, A B is therefore a lever of the first order; in the case of C D the force R′ is the power and W is the weight; C D is therefore a lever of the second order.

62. *The steelyard* is merely a lever A B weighted at the end B, near the fulcrum (F) on which it turns. At B is a hook or scale-pan, from which to hang the substance whose weight is required; there is a weight which can be moved backward and forward along the arm A F; when the arm A F is properly graduated, the weight of the substance in the scale-pan

FIG. 66.



is ascertained by the position of the moveable weight; and the graduations made for this purpose must be equal, as we shall proceed to show. It will be supposed that the lever exactly balances on the fulcrum when the moveable weight is taken off, and the scale-pan is empty, so that the weight of the instrument can be put out of the question. Suppose the moveable weight to be 1 lb., and that FA is seven times FB; if a mass is put into the scale-pan and balanced by 1 lb. at A, it is plain that the mass weighs 7 lbs.; for by taking moments about F we find that

$$(\text{weight of mass}) \times FB = (\text{weight of 1 lb.}) \times FA,$$

and we know that FA is seven times FB. Now divide FA into seven equal parts at *a, b, c, d, e, f*. Each part will be equal to FB; then the body must weigh 1 lb., if when placed in the scale-pan it is balanced by the moveable weight at *a*; it must weigh 2 lbs. if balanced by the moveable weight at *b*, and so on. Suppose, for instance, that the body were balanced by the moveable weight when placed at *d*; we have from the principle of moment

$$(\text{weight of body}) \times FB = (\text{weight of 1 lb.}) \times FD.$$

Now FD equals 4 . FB; and therefore the weight of the body must be 4 lbs, It is plain that if each division were subdivided into quarters, the weight of a substance could be ascertained to quarters of a pound by the one moveable weight, provided it did not exceed 7 lbs.

Suppose now it is required to ascertain the weights of substances with this instrument up to 1 cwt.; take three weights each of 4 lbs., three weights each of 1 lb., and the one moveable weight, these will be sufficient; each of the first three when placed at A will balance 28 lbs. at B; each of the second three will balance 7 lbs. at B, and the remainder will be ascertained by the position of the moveable weight. Thus, when a certain body is in the scale-pan, suppose that it is counterpoised by three of the former weights, two of the latter, and by the moveable weight midway, between *e* and *f*; the weight of the body must be $3 \times 28 + 2 \times 7 + 5\frac{1}{2}$ lbs., or 3 qrs. $19\frac{1}{2}$ lbs. Of course if the first three weights were marked 28 lbs., and the latter three 7 lbs., the weight of the body could be read off just as easily as when the weighing is performed by an ordinary balance. In this case the instrument enables us to weigh a hundredweight by weights amounting in all to 16 lbs. Any other division of FA might be adopted; e.g. if it were made long enough to admit of 28 principal divisions, it would enable us to weigh a hundredweight by a weight of 4 lbs.

We have assumed that when all weights are taken off, the instrument balances upon the fulcrum. This need not be the case; it is enough that the zero of the graduation should be at the point at which the moveable weight must be placed to balance the instrument when the scale-pan is empty. The student should prove this.

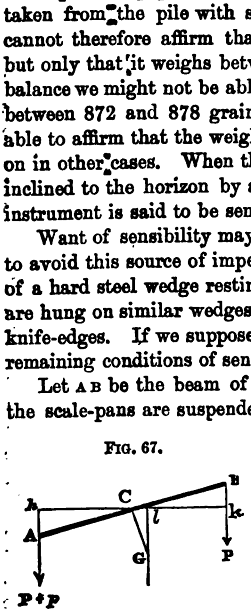
63. *The balance* is, in principle, merely a lever of the first order with equal arms, to each of which a scale-pan is attached, so that the whole

exactly balances on the fulcrum; it of course follows that if a mass is put into each scale-pan, and the whole continues in exact equilibrium, the masses must be equal. Nothing can be simpler; but in practice there is a difficulty which may be explained thus:—Take an ordinary balance such as is used for weighing letters, and put a 2-oz. weight in one scale-pan, and balance it by putting a pile of paper in the other; it will be found that a small piece of paper, weighing perhaps about a grain, can be added to or taken from the pile with scarcely any sensible effect on the balance; we cannot therefore affirm that the pile of paper weighs 2 ozs. (875 grains), but only that it weighs between 874 and 876 grains. With a less perfect balance we might not be able to affirm more than that the substance weighs between 872 and 878 grains; with a more perfect balance we might be able to affirm that the weight was between 874·9 and 875·1 grains, and so on in other cases. When the beam of a balance is caused to be distinctly inclined to the horizon by a very small difference between the weights, the instrument is said to be sensitive or to possess great sensibility.

Want of sensibility may be due to friction at the points of suspension; to avoid this source of imperfection, the balance is made to turn on the edge of a hard steel wedge resting on a hard smooth surface, and the scale-pans are hung on similar wedges—these wedges, we may remark, are often called knife-edges. If we suppose friction to be thus put out of the question, the remaining conditions of sensibility may be investigated as follows:—

Let A B be the beam of a balance, A and B the knife-edges from which the scale-pans are suspended, C the knife-edge on which the balance rests; let G be the centre of gravity of the beam. We will denote by W the weight of the beam, by $P+p$ and P the weights suspended at A and B, including the weights of the scale-pans; and by θ the angle at which A B is inclined to the horizon when the whole is in equilibrium. We will also denote the lengths of the equal arms A C and B C by a and C G by b. We have to determine the relation between these quantities. Through C draw the horizontal line $h\,k$, cutting the vertical lines through A, B and G, in h, k and l respectively. Now since the forces are in equilibrium about C, the principle of moments gives the equation

FIG. 67.



$$(P+p) \cdot C h = P \cdot C k + W \cdot C l.$$

But C h is equal to C k; we therefore have

$$p \cdot C h = W \cdot C l.$$

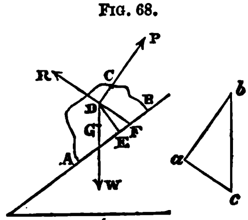
We might reason upon this equation, but it will be convenient to write it in a slightly different form. It is plain that A C h and C G l are equal to the angle we have denoted by θ, and consequently C $h = a \cos\theta$ and C $l = b \sin\theta$; the above equation is therefore equivalent to the following—

$$\tan\theta = \frac{p\,a}{W\,b}$$

It is plain from this that tan θ (and therefore also the angle θ) is greater for a given value of p, when a is greater, W less, and b less. Now for a given value of p, the sensibility is greater the greater the angle, and consequently the sensibility of a balance is increased, other things being the same, (1) by increasing the length of the arm, (2) by diminishing the weight of the beam, (3) by diminishing the depth of the centre of gravity of the beam below the point of support. These conditions to a certain extent conflict with each other; e.g. as the beam must not sensibly bend under the weight, it is not easy to lengthen the arm, without increasing the weight; again, if G is brought too near to C, the stability of the machine may be too much diminished. It is possible, however, to combine these conditions in such a manner that the instrument will render manifest a difference between the weights amounting to 1–500,000th of either; e.g. it will enable an experimenter to distinguish between a pound and a pound *plus* a seventieth part of a grain.

64. *Cases of a body resting against a fixed plane or planes.*—Before entering on this series of examples it will be only necessary to remind the reader that when there is one plane, its reaction is one of the forces which keep the body in equilibrium. If we suppose the plane to be smooth, the reaction will be exerted in a direction at right angles to the plane. If we suppose the plane to be rough, the direction will often be indeterminate, unless we know something more than merely that the body is at rest. This indeterminateness arises simply from deficiency of data; if we know that the body is on the point of sliding in some assigned direction, the direction of the reaction is then known. A similar remark applies when the body rests against two planes. The student should reconsider Arts. 10. 11, 12 before going further.

FIG. 68.



Ex. 52.—A body of given weight rests on a plane A B, inclined at a given angle to the horizon; it is acted on along a given line C P by a force (P); it is required to find P when on the point of making the body slide up the plane.

Draw the figure to scale and find G the centre of gravity of the body; through G draw the vertical line G W, cutting P C produced in D; as the forces P *and* W *are balanced* by the reaction (R) of the plane, that force

must act through the point D; draw D E at right angles to A B, and make the angle E D F equal to the angle of friction; the reaction must act along the line FD (as shown in the figure). That it must act along this line is plain—as the body is on the point of sliding, R's direction must make an angle with the perpendicular equal to the angle of friction, and as the sliding is about to take place in the direction A to B, F must be between E and B, as friction always opposes motion; if the sliding were about to take place in the opposite direction, F would be between E and A. The question is thus reduced to a case of the equilibrium of three forces acting on a point. Draw to scale $b c$ a vertical line proportional to W, and make $c a$ and $a b$ parallel to D R and D P, then $c a$ and $a b$ to the same scale will give the forces R and P.

Ex. 53.—Suppose the body to weigh 400 lbs., the plane to be inclined to the horizon at an angle of 20°, the line C P to be inclined to the plane at an angle of 10°, and the angle of friction to be that proper to wood on wood (18°). It will be found from a construction similar to that given in the last example that P and R are forces of 248·7 lbs. and 349·8 lbs. In other words, a force of 248·7 lbs. acting in the specified direction would just be enough to make the body slide up the plane; the mutual action between the plane and the body takes place along the line R F, and equals 349·8 lbs.; i. e. in consequence of the joint action of P and the weight, the body is urged against the plane by a force of 349·8 lbs. in the direction D to F, and the plane reacts against the body with an equal force in the opposite direction. In this determination there is only one point open to doubt, viz. the exact value of the angle of friction, about the magnitude of which we can say nothing more exact than this:—if the body were a mass of wood urged against a wooden plane, the angle would be about 18°, and if the angle were exactly 18°, the result arrived at in the present question would be strictly correct.

Ex. 54.—Other things being the same as in the last example, except that the mass and the plane are supposed to be both of metal, find the values of P and R.

In this case the angle of friction is 10°, and it will be found that P is a force of 200 lbs. and R of 346·4 lbs.

Ex. 55.—In the last example let it be required to determine the force which will just support the body on the inclined plane.

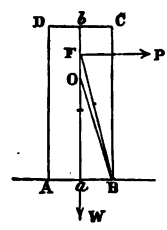
In this case, as the tendency of the body is to slide down the plane, the line F D R must be drawn on the side of D E, opposite to that shown in fig. 68, and (since in this case the angle of friction is 10°) making an angle of 10° with D E. The results arrived at will be that P is a force of 73·9 lbs. and R of 368·7 lbs. On comparing this result with that obtained in the last example the student will see that any force intermediate to 73·9 lbs. and 200 lbs. acting in the specified direction will support the body; the only effect produced by varying the force within these limits will be to change

the amount and direction of the mutual action between the body and the plane. Thus, if P were a force of 100 lbs., it would be found that R would be a force of 361 lbs., and acting along a line inclined to D E (fig. 68) at an angle of 6° 6′, and cutting the plane between A and E. On the other hand, if the plane were perfectly smooth, the mutual action would take place along the perpendicular E D (fig. 68), and with the same data as before it would be found that P will be a force of 138·9 lbs. and R of 351·7 lbs. If P were greater than 138·9 lbs., it would make the body slide up the plane; if less, it would not keep the body from sliding down. The student should verify these results, and should notice particularly the effect of the roughness of the plane in rendering equilibrium possible when the force P lies between certain limits; whereas, if the plane were smooth, equilibrium would be possible only when P has exactly one value.

One point connected with the previous examples deserves notice. It has been assumed that the point F falls within the base A B; if this were not the case, the body would be overthrown; e.g. suppose the force P to act along the line D C (fig. 68), and to be gradually increased, for every increase the direction of the mutual action is changed, and the point F will move towards B; if the body is shaped in such a manner that F reaches B before P has the value necessary to make the body slide, the body will be overthrown before it slides.

Ex. 56.—A cylinder of wood stands on a horizontal wooden floor, a horizontal force is applied to it and gradually increased; if the force is applied at a distance above the floor of more than $\frac{5}{3}$rds of the diameter of the base, the body will topple before it slides; if otherwise, the body will slide before it topples.

FIG. 69.



Draw the axis *a b*, and in it take a point O, such that the angle *a* O B may equal 18°; then *a* O will be nearly $3\frac{1}{4}$ times *a* B, or nearly $\frac{5}{3}$rds of the diameter A B. Let P be the force applied to the body, and let its line of action cut *a b* in F; join F B, and suppose P to be increased till the resultant of P and the weight (W) acts along F B; as the angle *a* F B is less than *a* O B, i.e. 18°, it is less than the angle of friction in this case, and consequently P is not large enough to make the body slide; but the smallest addition to P would overthrow it.

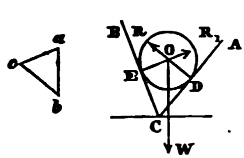
Ex. 57.—If the height of the cylinder in the last example were twice its diameter, and its weight were 100 lbs., a horizontal force of 25 lbs. applied to the body through *b* would be just sufficient to overthrow it; but if the horizontal force were applied so that its line of action cuts the axis at H, a point so chosen that *a* H equals *a* B, the body would not move unless the force were equal to 33 lbs., and then it would slide; in the latter case, if the force exceeded 100 lbs., it would cause both sliding and toppling.

Ex. 58.—If the plane is perfectly smooth and the force acts in a direction parallel to the inclined plane, show that the force, reaction, and weight are proportional to the height, base, and length of plane. This result, though easily proved, is of importance, and the student should make it out carefully; he should also notice that it follows that when a body is placed on a perfectly smooth horizontal plane the smallest possible horizontal force will move it.

Ex. 59.—A body is placed on a rough horizontal plane, and a line is drawn upward from the plane, making with it an angle equal to the angle of friction; show that this is the direction of the smallest force which will make the body slide.

Ex. 60.—A sphere or cylinder rests between two smooth, inclined planes; to find the pressure sustained by each plane.

FIG. 70.

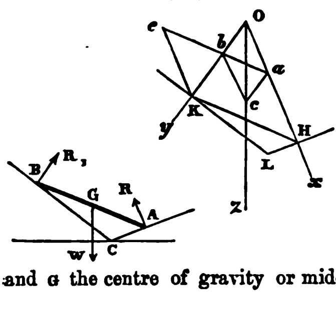


Let O be the centre of the sphere placed between the two planes CA and CB, and touching them at D and E respectively; if R and R_1 are the reactions of the planes, they will act along DO and EO respectively, and the three forces W, R, R_1 must be in equilibrium. Draw on scale the vertical line ab proportional to W, and draw bc and ca parallel to DO and EO; bc and ca will give on scale the magnitudes of R and R_1, which are equal and opposite to the pressures on the planes.

Ex. 61.—If CA and CB (on opposite sides of the vertical through C, as in fig. 70) are inclined to the horizon at angles of 50° and 70° respectively, and if W weighs 100 lbs., the reactions R and R_1 are 108·5 lbs. and 88·5 lbs. The pressures on the planes are equal and opposite to these reactions.

Ex. 62.—If CA and CB (Ex. 60) are on the same side of the vertical through C, and are inclined to the horizon at angles of 40° and 70° respectively (so that the angle between the planes is one of 30°), when the sphere weighs 120 lbs. the reactions will be respectively 225·5 lbs. and 154·3 lbs.; the pressures on the plane being equal and opposite to these forces.

FIG. 71.



Ex. 63.—A uniform rod is placed between two smooth inclined planes; find the position in which it comes to rest.

Let CA and CB be the planes, and G the centre of gravity or middle point of the rod AB. The rod is

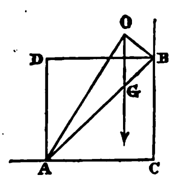
acted on by three forces, its weight acting vertically through G, and the reactions R and R_1 of the planes which act at right angles to CA and CB respectively; now when the rod is in its position of rest, the directions of these three forces must pass through a common point (Art. 41); the position can therefore be found as follows:—Make the angles $x\,o\,z$ and $y\,o\,z$ equal to the inclinations of CA and CB to the horizon; take $o\,c$ any length, and complete the parallelogram $o\,a\,c\,b$; draw the diagonal $a\,b$, and produce it till $a\,e$ equals the length of the rod; draw e K parallel to $o\,x$, and K H parallel to $a\,e$; draw L H and L K at right angles to $o\,x$ and $o\,y$; then if CA and CB were taken equal in length to L H and L K, A B would be the required position of the rod. The student should carefully make out that this construction gives the required position.

Ex. 64.—If the inclinations of CA and CB are 30° and 45° respectively, and if A B is a rod 8 ft. (96 in.) long, it will be found that when CA and CB are taken 41·8 in. and 76·3 in. long respectively, the rod will be in its position of equilibrium.

Ex. 65.—A rod (A B) with its centre of gravity in a given point (G) is placed at a given inclination with one end on the ground and the other against a vertical wall; ascertain whether it will continue at rest, the angle of friction (ϕ) between the lower end and the ground and that between the upper end and wall (ϕ_1) being given.

Draw the vertical and horizontal lines D A, D B; make the angle D A O equal to ϕ, and let A O cut the vertical line through G in O; join O B; if O B D is less than ϕ_1 the rod will continue at rest. There is nothing, however, in the data to show that the reaction of the ground at A acts along A O, or that the reaction of the wall at B acts along B O; in fact, with the data, these reactions and their lines of action are indeterminate. If O B D equals ϕ_1, the rod is just on the point of sliding. In this case the directions, and therefore also the magnitudes, of the reactions are determinate.

FIG. 72.



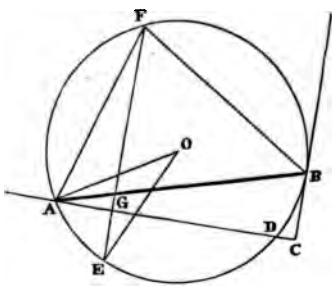
Ex. 66.—If the angles of friction at A and B are each 33°, find how far from the end A of the rod its centre of gravity must be if it stand just on the point of sliding when inclined at an angle of 45° to the horizon; the length of the rod being 40 ft. A diagram must be drawn like fig. 72; the angles B A C, D A O, D B O, being made equal to 45°, 33° and 33° respectively; a vertical line drawn through O will cut the line A B in G the centre of gravity, which will be found to be slightly more than 30 ft. from the end A.

Ex. 67.—With the data of Ex. 65, find in what position the rod will slide.

Draw A B the rod and mark G its centre of gravity; on A B describe a segment of a circle containing an angle equal to $90°+\phi-\phi_1$. At O the centre of the circle make the angle A O E equal to 2ϕ; join G E; if A G E

equals or exceeds a right angle, the rod will not slide in any position; if AGE is less than a right angle, draw AD at right angles to GE, then BAD is the inclination of the rod to the horizon when on the point of sliding. We can easily prove this, for if we draw BC at right angles to AD, the relative positions of the rod, the ground, and the wall, when the rod is on the point of sliding ought to be given by AB, AC, and BC; and that this is the case is evident, for join AF and FB, the angle AFE = $\frac{1}{2}$ AOE $=\phi$, so that EFB $=90-\phi_1$, i.e. CBF $=90+\phi_1$. If the reactions come along these lines they will balance the weight, and as they act along these lines, the points A and B are both in the act of sliding, by what was shown in Ex. 65.

FIG. 73.



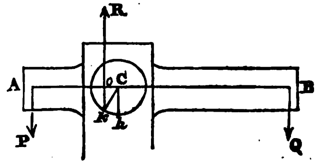
Ex. 68.—If the angles ϕ and ϕ_1 are equal (i.e. ground and wall equally rough), and if the rod is of uniform density so that its centre of gravity is at its centre, the rod will begin to slide at an inclination of $90-2\,\phi$ to the horizon.

Ex. 69.—If AB is 40 ft. long, and its centre of gravity 10 ft. from A, and if the angle of friction at A is 18° and that at B 38°, it will be found that the rod will begin to slide when inclined at an angle of 19° 51′ to the horizon.

65. *Body in equilibrium on a rough axle.*—It is usual to suppose that the axle may be treated as a geometrical line, or, which comes to the same thing, that there is no friction between the axle and its bearing; it will be instructive, however, to consider a case in which the friction is taken into account.

Ex. 70.—Let AB be a lever of the first order, capable of turning on an axle C of given radius, the co-efficient of friction between which and its bearing is known; required the relation between the forces P and Q—supposed to act vertically downward—when P is on the point of preponderating. Draw the vertical radius Ch, make the angle hCk equal to the angle of friction, and draw kR a vertical line cutting AB in O.

FIG. 74.



When P *is on the* point of making the axle slide on its bearing, the re-*action of the bearing*, and therefore the resultant of P and Q, will act along

k R. This is evident; for if the reaction acts from k to R, the friction will act in the direction from h to k, i.e. in the opposite direction to that in which sliding is on the point of taking place. Now, since O is a point in the resultant of P and Q, and the forces tend to turn the body in opposite directions, the moments of P and Q with respect to O must be equal, and therefore O A . P equals O B . Q. The distance C O may be determined by drawing the axle to a large scale; but it can be found with sufficient accuracy as follows: let r denote the radius of the axle and μ the co-efficient of friction between axle and bearing; C O is strictly equal to $r \times$ sine of angle of friction, and this, at all events when an unguent is used, will not sensibly differ from $r\mu$. If, therefore, C A and C B are denoted by p and q respectively, O A equals $p - r\mu$ and O B $q + r\mu$; so that we obtain the relation

$$P(p - r\mu) = Q(q + r\mu).$$

If Q were the preponderating force, O would fall between B and C, and we should obtain the relation

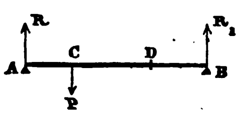
$$P(p + r\mu) = Q(q - r\mu).$$

Ex. 71.—Let the lever be a mass of 5 tons whose centre of gravity is 15 in. from the centre of the axle, and let P act at a distance of 4 ft. from the same point; also suppose the radius of the axle to be 6 in., and the coefficient of friction between axle and bearing to be 0·1. Then P, when on the point of overcoming the weight, must be a force of 3686 lbs., and when it only just prevents the weight from falling, it must be a force of 3319 lbs. If P has any magnitude between 3319 and 3686 lbs. it will support the weight. If the axle were smooth, P must be a force of exactly 3500 lbs. to balance the weight. Two other points call for notice: (1) Suppose P to change gradually from 3319 to 3686 lbs., the weight may be slightly raised during the change in virtue of the *rolling* of the axle on the bearing, provided the radius of the axle is smaller than that of the bearing. (2) The axle will, in reality, rest on two bearings, as in the case of the trunnions of a gun; one bearing supporting one half of the whole force P + Q, while the other bearing supports the other half.

66. *Applications of couples.*—In the following examples use will be made of the properties of couples; the student will therefore find it advantageous to go carefully through Art. 51–55, before considering what follows.

Ex. 72.—A rod A B without weight rests on two points, one under each end; a given weight (P) is suspended from a given point of it (C); find the tendency of the weight to break the rod at any specified point (D) between C and B.

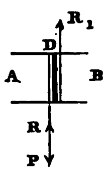
FIG. 75.



The rod is under the action of three forces, P and the reactions R and R_1 of the fulcrums A *and* B. *Assuming* that the rod *bends but slightly,* R will equal P × B C ÷ A B, and R_1 will equal P × A C ÷ A B.

Now the force R_1 acting at B is equivalent to a force R_1 acting at D in the same direction, together with a couple whose moment is $+R_1 . BD$; in the same manner the forces R and P, acting at A and C, are equivalent to equal forces acting in the same direction at D, together with couples whose moments are $-R . AD$ and $P . CD$. Let fig. 76 represent a portion of the rod in the neighbourhood of the point D. We see in the first place that the tendency of the transferred forces is to divide the rod by making the parts AD and BD slide in opposite directions over the line at D; since $R+R_1$ equals P, the tendency to make DB slide upward is equal and opposite to the tendency to make DA slide downward. When two forces have this tendency to make two parts of the same body slide over each other in opposite directions, they are said to be shearing forces; and the tendency to produce shearing may be measured by either of them, in the same way that either of the equal forces which stretch a thread measures its tension. In the next place the tendency of the one couple $+R_1 . DB$ is to turn DB round D in the opposite direction to that of the motion of the hands of a watch; that of the other couples (which are equivalent to a single couple $-R . DA + P . DC$, and this to a couple $-R_1 . DB$) is to turn DA round D in the same direction as that of the motion of the hands of a watch. If the rod is sufficiently strong, these two tendencies neutralize each other, and the rod is slightly, perhaps imperceptibly, bent. The measure of this tendency is the moment of either couple, which is therefore called the bending moment. Since R_1 equals $P \times AC \div AB$, the bending moment at D plainly equals

FIG. 76.



$$\frac{P . AC . BD}{AB}.$$

On the whole, therefore, the forces tend to break the rod at D in two ways, (1) by shearing the rod across at D, (2) by bending the rod at D. The shearing force equals $P \times AC \div AB$ at all points of the rod between B and C. The bending moment equals $P \times AC \times DB \div AB$ at D, and increases from zero at B to $P \times AC \times CB \div AB$ at C, where it has its greatest value.

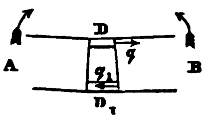
Ex. 73.—If the lengths of AB and AC are 12 ft. and 4 ft. respectively, and P is a weight of 180 lbs., the shearing force at D is 60 lbs. and the bending moment $60 \times BD$; the greatest value of the bending moment is at C, where it equals 480, feet and pounds being taken as units; if BD were 3 ft., the bending moment would be 180, or only $\frac{3}{8}$ths of that at C.

Ex. 74.—If we take E, any point between A and C, the shearing force at E is $P \times BC \div AB$, and the bending moment $P \times BC \times AE \div AB$; the latter having for its greatest value $P \times BC \times AC \div AB$, the same as that found in Ex. 72.

67. *Remark.*—The nature of the resistance offered by the rod to the bending moment will be readily understood by considering the following *case;—Suppose the* rod to be completely separated at D, and the parts

united by two short pieces of any strong and elastic substance (as shown at D and D_1 in fig. 77, where, for convenience, the thickness of the rod is exaggerated). As already explained, the effect of the couples is to bend the rod slightly, and consequently to compress D and stretch D_1: let us denote the distance $D D_1$ by b, and the resistance offered by D to compression by q, and that offered by D_1 to extension by q_1. Then the reactions of D and D_1 against D B will be q and q_1, as shown in the figure; these two forces balance the couple $R_1 . D B$, and consequently must form a couple of equal moment, i.e. q must equal q_1, and

FIG. 77.



$$b q = R_1 . DB = \frac{P . AC . BD}{AB}.$$

Thus, if we take the data in Ex. 73 and suppose the rod to be 1 in. ($\frac{1}{12}$ ft.) thick, we see that when D is 3 ft. from B $\frac{1}{12} \times q = 180$, or $q = 2{,}160$ lbs., i.e. D_1 is stretched and D compressed by forces of 2160 lbs; and in like manner, if D were taken at C (fig. 75), q would equal 5760 lbs. Of course in the actual case the matter between D and D_1 is continuous, and each portion bears its part in resisting the bending moment; but this example is sufficient to show what very large forces must be called into play to resist the bending moment when the thickness of the rod is small in comparison with its length. It accordingly happens in the case of rods and beams that the shearing force is of but little importance in comparison with the bending moment; but when the rod is short and the forces applied to it are large, the shearing force becomes of most importance. This is the case with the rivets by which boiler plates are joined; the forces which they have to resist are shearing forces.

Ex. 75.—In a pair of pincers the jaws meet at $1\frac{1}{5}$ in. from the rivet; the handles are grasped with forces of 50 lbs. at a distance of 6 in. from the rivet; the rivet will have to resist two shearing forces each of 250 lbs.

Ex. 76.—Let the rod in Ex. 72 be of uniform density, and unloaded except by its own weight; the shearing force and bending moment at D will be $\frac{1}{2} W \left(1 - \frac{x}{a}\right)$ and $\frac{1}{2} W x \left(1 - \frac{x}{2a}\right)$; where W denotes the weight of the rod, $2a$ its length, and x the distance of D from A.

Ex. 77.—A weight P hangs from a point (D) connected by two rods A D, C D, with a third rod A B, which rests in a vertical position with the end B on the ground, and is kept from turning over by two fixed points E and F; required the pressures on the points supposed to be smooth.

Let the perpendicular distance of D P from A be denoted by a, the reactions of E and F by R and R_1, and the distance E F by b; as there is no

friction between the points and the rod, the forces R and R_1 must act in a direction at right angles to A B. The force P acting at D is equivalent to a force P acting along A B and a couple whose moment is a P, the former force is wholly supported by the ground, the couple is balanced by the forces R and R_1, which must therefore form a couple of equal moment and opposite sign; consequently R must equal R_1; the forces must act as shown in the figure, and b R must have the same magnitude at a P, i.e. R must equal $a\,P \div b$. Suppose, for instance, that the weight of P is three tons, that P D is horizontally 15 ft. distant from A, and that E and F are $2\frac{1}{2}$ ft. apart, the weight of three tons will have to be borne by B, and the points E and F will sustain horizontal pressures of 18 tons apiece; these pressures will be equal and opposite to the reaction shown in the figure as S, R and R_1. If we take into account the weight of the rods, nothing new would be introduced in principle; it would be equivalent to a weight acting along A B and a couple. This case resembles that of the equilibrium of a loaded crane, and it will be observed that the solution is obtained without reference to the actual forces transmitted along A D and C D. If we take A C and A D to be 4 ft. and 25 ft. long, and therefore C D to be 28·3 ft. long, and suppose that the joints A, C and D are without friction, the forces (T and Q) transmitted along A D and C D will be found, as in Ex. 50, to equal $18\frac{3}{4}$ tons and $21\frac{1}{4}$ tons. Consequently in this case the rod A B is acted by the five forces R, R_1, S, T, Q shown in the figure. In practice these results might not be exactly correct, as they would be to some extent modified by the friction at the points E and F, and by partial rigidity at the joints A, C, D. There are ordinarily no data for determining the extent of these modifications; and in designing any structure of this kind it would be assumed that A D sustains the tension of $18\frac{3}{4}$ tons, C D the thrust of $21\frac{1}{4}$ tons, E and F the pressures of 18 tons, and A B the five forces indicated in the figure, and the parts would be made sufficiently strong for that purpose.

FIG. 78.

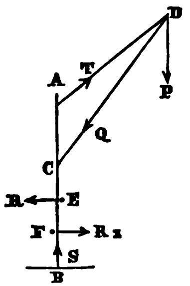
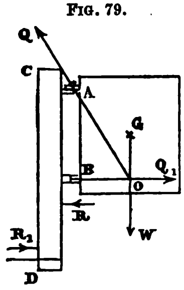


FIG. 79.

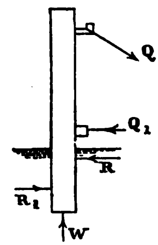


Ex. 78.—A swing gate rests on a hinge and against a turning-point; what are the conditions of its equilibrium?

Let A be the hinge and B the turning-point; G the centre of gravity of the gate, whose weight is W; the horizontal distance of G W from A or B we will denote by a. Now ultimately W must be supported by the ground, and as W acting along G W is equivalent to an equal force W acting along the gate-post, and a couple whose

moment is a W tending to turn the post in the same way as the hands of a watch move, the reaction of the ground against the lower part of the post must be such as to produce an upward force equal to W, and two equal horizontal forces such as R and R_1, forming a couple whose moment will equal a W; but we cannot say exactly how much of the reaction of the ground is exerted at each point of the part of the post that is within the ground. In the next place, the force must be transmitted in some way through the hinge and the turning-point. To ascertain the forces exerted at these points we may reason thus:—W is supported by reactions at A and B; that at B, which we will denote by Q_1, will be exerted horizontally; the reaction of the hinge (Q) must be exerted in such a manner and be of such an amount that it will balance W and Q_1. Let the horizontal line through B cut G W in O; join A O; Q must act along O A as shown in fig. 79. A B O is a triangle fulfilling the condition of the triangle of forces, and consequently Q : W :: A O : A B and Q_1 : W :: B O : A B, i.e.

FIG. 80.



$$Q = \frac{a\,W}{\sqrt{a^2 + b^2}} \text{ and } Q_1 = \frac{b\,W}{a}.$$

The forces thus found are the reactions of the hinges against the gate.

The actions of the gate on the hinge and turning-point are of course equal and opposite to the reactions; if then we put the weight of the gate-post out of the question, the forces acting on the post are those shown in fig. 80. The forces R, R_1 and W are to a certain extent indeterminate as above mentioned; this indeterminateness arises from want of sufficient data. The resistance offered by the cohesion and friction of the earth at various points is unknown; only its total effect is known.

Ex. 79.—If the distance between the hinge and the turning-point is 4 ft., the horizontal distance of centre of gravity of gate from hinge 5 ft., and weight of gate 112 lbs., the pressure on the turning-point will be 140 lbs., and the pull on the hinge 179 lbs., the inclination to the horizon of the line along which the latter force acts being 38° 40′.

Ex. 80.—In the last case, if the gate is wide enough to allow a weight of 100 lbs. to hang on it at a horizontal distance of 10 ft. from the hinges, there will be an additional horizontal pressure on the turning-point of 250 lbs., and an additional pull of 270 lbs. on the hinge exerted at an angle of 21° 48′ to the horizon.

[If the gate is allowed to swing, the centrifugal force of the gate will increase the pull on the hinge, but diminish the pressure on the turning-point.]

Ex. 81.—If the gate hangs on two hinges, instead of a hinge and a turning-point, there will be a horizontal pull on the upper hinge, and a horizontal pressure on the lower hinge, each force being equal to $a \text{w} \div b$; there will also be vertical pressures on the two hinges, together amounting to W; the amount of each separately is unknown. This indeterminateness arises from the fact that, with the data, the lines along which the reactions of the hinges are exerted are unknown, except that they must intersect somewhere in the vertical line drawn through the centre of gravity of the gate.

QUESTIONS.

1. What is meant when a line is said to represent a force?

2. Represent on paper two forces of 40 and 50 units respectively acting along lines inclined at an angle of 37°. Assuming the point of intersection and the position of one line to be given, in how many different ways may the forces act in conformity with the above directions?

3. State the principle of the parallelogram of forces.

4. Draw two lines, O A, O B, containing an angle of 120°; let a force of 12 units act from O to A, and one of 10 units from O to B; find their resultant.

5. Find the resultant in the last question, supposing the direction of the force of 10 units is reversed.

6. Define the components and rectangular components of a force.

7. Through a point O draw four lines, O A, O B, O C, O D, in order such that A O B is an angle of 30°, A O C of 90°, A O D of 120°; a force of 8 units acts on O from O to B; find its components, (1) along O A and O C, (2) along O A and O D, (3) along O C and D O produced.

Ans. (1) 6·93 and 4, (2) 9·24 and 4·62, (3) 16 and 13·86.

8. State the condition of equilibrium of three forces acting on a point.

9. Draw a square A B C D; a force of 5 units acts on A from D to A, one of 3 units from A to B; find (1) their resultant, (2) the force which balances them.

10. State the principle of the triangle of forces.

11. Three forces of 12, 16, and 20 units respectively act on a point; show how they must be adjusted when in equilibrium.

12. State the conditions of equilibrium of three forces acting on a rigid body.

13. A rod (A B) without weight can turn freely round a fixed point or hinge at one end (B); it is held in a horizontal position by a force (Q) of 50 *units*, which acts vertically downward through its middle point, and by a

force (P) which acts at the end A in such a manner that the angle BAP equals 30°. Determine P and the pressure on the fixed point.

Ans. There are two cases; in both P and pressure on fixed point are 50 units apiece.

14. Suppose that instead of the last question we had the following:— A rod of uniform density (AB) weighing 50 lbs. can turn freely round a fixed point or hinge at the end B; it is held in a horizontal position by a force P acting through A along a line which makes an angle of 30° with AB; find P when it will just support the rod, and the pressure on the fixed point. Why would these two questions be in effect the same?

15. What is meant by the moment of a force with respect to a point? What does the moment measure? What is the meaning of the signs + and − when prefixed to moments?

16. Draw a square ABCD whose side is 4 in. long; let a force of 7 units act from A to B, one of 9 units from C to B, one of 3 units from D to C, and one of 5 units from A to D; write down with their proper signs the moments of these forces, (1) with respect to the point of intersection of the diagonals, (2) with respect to the angle A.

Ans. If the letters ABCD follow each other in the opposite direction to that of the motion of the hands of a watch, (1) +14, −18 −6, −10; (2) 0, −36, −12, 0.

17. State the principle of moments.

18. In both cases of Q. 16 find the moment of the resultant of the forces.

Ans. (1) −20, (2) −48.

19. Prove the principle of moments in the case of two intersecting forces and their resultant.

20. When a plane can move freely round a fixed point and is kept at rest by two forces acting in that plane, what relation exists between the two forces? Strictly speaking, a third force acts on the plane; what is that force?

21. Draw a square ABCD and take E the middle point of AD; suppose a force (P) of 5 units to act from A to B and a force (Q) of 17 units from D to C; (1) what force (R) acting along BC will balance these forces, E being a fixed point? (2) what force besides P, Q, R acts on the square? (3) find the force R if A were fixed instead of E.

Ans. (1) 6 units from B to C, (3) 17 units from B to C.

22. When are two parallel forces said to act in the same and when in opposite directions? Give the rules for finding the resultant of two parallel forces.

23. Draw a square ABCD; a force of 6 units acts from A to D, and one of 4 units from B to C; find their resultant. Also find their resultant when the direction of the force of 4 units is reversed.

24. In the last question, let a force of 25 units act from D to A, and one of 24 units from B to C; find their resultant.

25. Give at full length the reasoning by which both the rules of Art. 46 are proved.

26. Show that the principle of moments is true of two parallel forces and their resultant. Consider the case in which P and Q act in opposite directions, Q being greater than P, and O between A and B (Art. 48).

27. State the conditions of equilibrium of three parallel forces.

28. A weightless rod of indefinite length can turn freely round at point C; a point A of the rod is taken 4 ft. from C. At right angles to the rod and through A acts a force P of 12 units, a parallel force Q of 16 units acts through another point B. Find the position of B and the reaction of the fixed point, (1) when P and Q act in the same, (2) when they act in opposite directions.

29. What is meant by the centre of two parallel forces? In making the statement consider the case in which the forces act in contrary directions as well as that in which they act in the same direction.

30. Define a couple, its arm, and its moment. In what respect do two equal parallel forces acting in opposite directions differ from every other combination of two forces acting in one plane?

31. Draw a square ABCD; a force of 12 units acts from A to D, and an equal force from C to B; what is the moment of the couple? Suppose the former force to be increased by 0·01 of a unit; find the resultant of the forces.

32. Prove that two couples of equal moments and of contrary signs acting in one plane on a rigid body are in equilibrium.

33. Draw a straight line AB, 8 in. long, and mark its middle point C; forces of 12 and 3 units act upward at A and C at right angles to the line, at B a parallel force of 12 units acts downward; find the force that will balance them.

34. Two couples acting in the same plane are equivalent to each other; in what respects must they agree, and in what may they differ?

35. How is the resultant of any two couples found?

36. Draw in any position on your paper two squares, ABCD, EFGH, and let AB be an inch long and EF three inches long. Suppose forces of 2 units to act from A to D and C to B, and forces of 20 units from A to B and C to D; what forces acting along EF and GH will be equivalent to the four forces?

37. What result is obtained by compounding a force with a couple? Draw a square ABCD; forces of 6, 12 and 12 units act from A to B, B to C and D to A; find their resultant.

38. Let A and B be two points 10 ft. apart, such that the line joining them is inclined at an angle of 30° to the horizon; a thread 15 ft. long

has its ends fastened to A and B; to the middle point of the thread a weight of 50 lbs. is tied; find the tensions of the two parts of the thread.

Ans. 47·8 lbs. and 10 lbs.

39. In the last question, if the weight were fastened to a smooth ring through which the thread passes, find the position in which the whole comes to rest, and the tension of the thread.

Ans. (1) 35° 16′ inclination of thread to vertical; (2) tension 30·6 lbs.

40. A weight (B) of 80 lbs. is suspended by a thread from a point A; the weight is pulled horizontally by a force P, such that when the weight comes to rest A B is inclined at an angle of 60° to the vertical; find P and the tension of the thread. *Ans.* 138·6 lbs., 160 lbs.

41. In Ex. 38 (p. 64) suppose D C and A B to be inclined at angles of 60° and 10° to the vertical, and B C to the horizontal; if W is a weight of 100 lbs., determine W_1 and the tensions of the parts of the thread. What are the forces which act on C D?

Ans. W_1 equals 10·18 lbs.; tensions on A B, B C, C D are 101·5, 17·6, and 20·36 respectively.

42. State the suppositions on which questions relating to tensions and thrusts of rods are solved.

43. Two equal rods (as in Ex. 44) rest with their feet against points B and C in the same horizontal line; each rod is 12 ft. long; B C is 20 ft.; a weight of 1000 lbs. is hung from the point A; what are the forces transmitted along the rods to B and C respectively? *Ans.* 904·5 lbs.

44. In the last question, assuming that the structure stands, what are the forces exerted on each rod?

45. In Ex. 46 in what ways do the forces tend to break the rods A B, B C, C A respectively? (Art. 35, *e.*)

46. Adapt the treatment of Ex. 46 to the case in which P, Q, R are parallel forces, and obtain numerical results when A B, B C, C A are 12, 20, 16 ft. respectively, and the force at A 250 lbs. acting at right angles to B C. (Fig. 62.)

Ans. Q and R equal 160 and 90 lbs.; thrusts of A B and A C and tension of B C, 200, 150 and 120 lbs.

47. Explain the use of the tie-beam of a roof (Ex. 49).

48. The rafters and tie-rod of a roof form an equilateral triangle; it is known that the weight has the same effect as if a weight of 8000 lbs. were hung from the roof-tree. If the tie-rod is of iron, what must be its cross-section to sustain with safety the tension it undergoes? (Art. 32.)

Ans. 0·4 sq. in.

49. In Ex. 50, what ought to be the section of the rod A C—supposed to be of iron—to support the tensile stress with safety?

Ans. 6·72 sq. in.

50. Distinguish between the three kinds of levers. Give an example of each kind.

51. Explain the principle of the common steelyard. What advantage is obtained by its use?

52. In fig. 66, if A F is 28 times as long as B F, and is divided into 112 equal divisions, and if a moveable weight of 1 lb. is used; what weights would be necessary to counterpoise a body weighing $601\frac{1}{4}$ lbs., and how would they be placed?

53. Illustrate the difficulty of exact weighing, and state what is meant by the sensibility of a balance. What is meant by a knife-edge? Why are knife-edges used?

54. Putting friction out of the question, investigate the conditions of the sensibility of a balance.

55. Draw a square A B C D, and suppose it to represent a cube of stone, weighing 1000 lbs., and resting on a stone floor (A B produced); through D suppose a force P to act tending to make the body slide in the direction B to A. If P will just make the body slide, find its magnitude, supposing that A D P is (*a*) an angle of 90°, (*b*) of 110°; (*c*) find the magnitude and direction of P when it is the smallest force that will make the body slide; (*d*) find the magnitude of the mutual action between the plane and the body in the previous cases; (*e*) suppose the angle A D P to be 120° and P to be a force of 200 lbs., find the magnitude and line of action of the mutual action between plane and body (Arts. 11, 12).

Ans. (*a*) 650 lbs.; (*b*) 559 lbs.; (*c*) 544·6 lbs., A D P equals 123°; (*d*) 1192, 964·4, 838·7 lbs.; (*e*) 916 lbs., inclined to the vertical at an angle of 10° 54, and cutting A B at a point distant 0·36 A B from A.

56. Explain from Ex. 65 why, when a man walks up a ladder, it becomes more likely to slip as he gets nearer the top.

57. A fly-wheel 10 ft. in radius weighs 20 tons, its axle is 18 in. in diameter, coefficient of friction between it and its bearing 0·15; a band passes round its circumference, from the end of which hangs a weight (W); find the magnitude of W when it will just turn the wheel.

Ans. 510 lbs.

58. In Ex. 70 the lever is supposed to be of the first kind; suppose it to be of the second or third kind, i.e. P to act upward at some point in C B produced or C B; show that when P is the preponderating force $$P(a+\mu r)=Q(b+\mu r).$$

59. In the last question given that the values of μ, r, a, b, and Q are 0·1, 6 in., 2 ft., 10 ft., and 2000 lbs. respectively; show that equilibrium can be maintained by any value of P acting vertically upwards between 392 lbs. and 408 lbs.

60. In Ex. 76 show that at the middle point the shearing force is 0, and the bending moment $\frac{1}{4}$ W a.

61. In the last Question, if W were concentrated at the middle point of the rod, show (Ex. 72) that the shearing force is $\frac{1}{2}$ W and the bending moment $\frac{1}{2}$ W a.

62. In Ex. 76, suppose W to equal 2000 lbs., a to be 10 ft. long; show that the bending moments at the middle point and at points distant 5 ft. from each end are respectively 5000 and 3750.

63. Four planks equal in all respects are placed flat on one another, and rest with their ends on two trestles, it is found that a weight W bearing on their middle point will be sufficient to break them; if the four planks are put together so as to form a hollow box open at both ends, it will be found that they will now support W; account for this. (See Art. 67.)

64. Let the rod in Ex. 76 be loaded not only by its own weight (W), but also by a weight (W_1) acting at a point C (fig. 75); if b and x denote BC and BD, show that the bending moment at D equals

$$\tfrac{1}{2}\,W\,x\left(1-\frac{x}{2a}\right)+W_1\,x\left(1-\frac{b}{2a}\right).$$

65. In the last Question let W, W_1, $2a$, b equal 1000 lbs., 400 lbs., 20 ft., 15 ft.; show that the bending moment at a point between B and C, and at a distance x from B, is $600x-25x^2$, and that it has its greatest value at a point distant 12 ft. from B.

N.B.—The Student will observe that the references to *Examples* are to those that occur as parts of the several chapters, and are marked thus—Ex. 75, Ex. 101, &c. The references to *Questions* are to those added to the ends of the chapters under the heading of Questions.

CHAPTER IV.

WORK.

68. *Definition and measure of work.*—When the point of application of a force moves wholly, or partly, in the direction of the force, work is said to be done by the force. Suppose a force of P units to act on a point, and suppose the point to move in the direction of the force, and to describe a distance of p units; the work done by the force while its point of application describes this distance is measured by the product P p. A force of one unit does a unit of work when its point of application moves through a unit of distance in the direction of the force; consequently the product Pp is the number of *units of work* done by P. When force is measured in pounds and space in feet it is sometimes convenient to call the unit of work a 'foot-pound.'[1] Thus if a carpenter urges forward a plane through 3 ft. with a force of 12 lbs., he does 36 foot-pounds of work; or, if a weight of 7 lbs. descends through 10 ft., gravity does 70 foot-pounds of work.

If in virtue of the action of other forces the point moves in a direction opposite to that of the force, the force resists the motion of the point, and work is expended in overcoming that resistance, or the work is said to be done against the force. If the force is one of P lbs. and the point moves through p ft. in a direction

[1] When there is occasion to draw attention to the fact that distance is assumed to be measured in feet and force in pounds, it is best to use the term foot-pound; under other circumstances it is, perhaps, best to use the more general term 'unit of work.'

opposite to that of the action of the force, P p units of work will be expended in overcoming the resistance of the force. Thus, if a weight of 10 lbs. is raised to a height of 5 ft., 50 foot-pounds of work must have been expended in overcoming the resistance of gravity, i.e. 50 foot-pounds of work have been done against gravity.

It is not necessary that the point should move in a straight line. The above statements are true without any modification, when the point moves in a curved line, provided the direction of the force is always tangential to the curve. Thus, if a man presses with a force of 30 lbs. at right angles to the arm of a capstan, and if, in one turn of the capstan, the point at which he pushes describes a circumference of 60 ft., he does 30×60, or 1800 foot-pounds of work.

Ex. 82.—If the area of the piston of a steam-engine is 5000 sq. in., and the mean pressure of the steam is 12 lbs. per sq. in., the whole force will be 60,000 lbs.; and the work done by the steam in one stroke of 8 ft. will be 480,000 foot-pounds.

Ex. 83.—If two weights of 150 and 200 lbs. are raised through heights of 80 and 120 ft. respectively, the whole work done must be $150 \times 80 + 200 \times 120$, i.e. 36,000 foot-pounds. It is plain that this is the same number of units as must be done if 350 lbs. were raised through $102\frac{6}{7}$ ft. Now the student will easily prove that when the weights are raised as in the Question, their centre of gravity will be raised through $102\frac{6}{7}$ ft. So we see that the number of units of work done in raising the weights separately is the same as the number that would be done if the whole weight were raised through the same height as that through which the centre of gravity is raised.

69. *Work of raising a system of weights.*—The result exemplified in the last Example is perfectly general and may be stated thus:—*When two or more heavy points are raised through different heights, the work done equals the sum of the weights of the points, multiplied by the height through which their centre of gravity has been raised.* Two remarks may be made on this principle. *First*, if the weights are the parts of a continuous body,

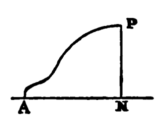
the work expended in lifting it is the product of the whole weight multiplied by the vertical height through which the centre of gravity is raised. This is equally true whether it is raised as a whole or in parts; e.g. the work done in lifting the earth when a well is sunk through a stratum of uniform density is the weight of the earth multiplied by half the depth of the well. *Secondly*, if a body moves in such a way that its centre of gravity neither rises nor falls, work is neither expended on gravity nor done by gravity. If work is expended on the motion at all, it is expended on friction or on some force other than gravity; e.g. when a train moves on horizontal rails, work is expended only on friction, resistance of the air, &c., none on the weight of the train.

Ex. 84.—A well-shaft is 300 ft. deep and 5 ft. in diameter; it is full of water; how many units of work must be expended in getting this water out of the well (i.e. irrespectively of any other water flowing in)?

The weight of the water is $\pi \times (2 \cdot 5)^2 \times 300 \times 1000 \div 16$ or 36,815 lbs. The centre of gravity is 150 ft. below the ground; consequently the work expended must be $368{,}155 \times 150$, or 55,223,250 foot-pounds.

70. *Cases in which the point of application does not move along the line of action of the force.*—Such a case arises when a weight falls obliquely. Thus, let A N be a horizontal line, and suppose a weight to fall from P to A along the curved line; draw the vertical line P N. The work done by gravity is the weight multiplied by P N.

Fig. 81.



Since gravity acts on the body along a vertical line throughout the whole motion, P N is the distance through which gravity has acted, measured in the direction of the force. Other cases can be similarly treated, and thus we may say generally that the work done by a force equals the product of the force, and the distance through which its point of application moves, *measured in the direction of the force.* Of

course, if the weight is lifted from A to P, the work expended equals the weight multiplied by N P.[1]

Ex. 85.—A train weighing 100 tons is made to run up an incline a mile long of 1 vertical to 160 horizontal; the train is lifted through $5280 \div 160$, or 33 ft. of vertical height; consequently $224{,}000 \times 33$, or 7,392,000 foot-pounds, must be expended.

71. *Cases in which a force does no work.*—It is very possible for a force to act on a moving body and to do no work. This is evident; for in order that a force may do work, its point of application must move wholly or partly in its direction. Suppose, then, that the point of application of the force is at rest, or that it moves in such a manner that no part of the motion is in the direction of the force; in either case the force does no work. Thus:—

(*a*) Suppose a point to slide along a smooth horizontal plane; the only forces acting are its weight and the reaction of the plane; both act at right angles to the direction of the motion, so that no part of the motion is in the direction of the forces, and consequently neither force does work, nor has work expended on it. In actual cases of sliding there is, in addition to the above-named forces, a force of friction which acts horizontally in a direction opposite to the motion, and as long as motion lasts work must be expended in overcoming it.

(*b*) Consider the case of a wheel rolling along a road. If we suppose this to be the case of a perfectly hard circle rolling (without any sliding) on a perfectly hard straight line, it is plain that at each instant the circle touches the line at one point only, and that point is for the instant at rest.

[1] If the space through which the point of application of the force moves is indefinitely small, the work done by the force is what is generally called its *virtual moment*, and may be reckoned positive; in a similar case, if work is expended on the force, this work is the *virtual moment* of the force, and is reckoned negative. Suppose now the space through which the point moves to be finite; divide the path into an indefinitely great number of parts, let P denote the force, and p any one portion of the path measured in P's direction, i.e. P's virtual velocity; and let the sum of all the products Pp be taken for all the parts of the path; this sum is the work done by P. It will be observed that this statement includes the cases in which P is variable, and in which the virtual moment of P is positive in some part of the path and negative in others.

Now the reaction of the road on the wheel acts through the point of contact, and consequently does no work. The above suppositions are very commonly made in solving mechanical questions; in reality, however, the wheel compresses the road, and the resistance which has thus to be overcome is what mainly renders an expenditure of work necessary to keep the wheel rolling.

(*c*) Consider the case of a body turning upon an axis supported by a fixed bearing, e.g. a fly-wheel, or a wheel and axle. If we suppose the axis to be a geometrical line, the reaction of the bearing acts on a point which has no motion, and consequently does no work nor has work expended on it. If we suppose that the axis has a sensible diameter, but that there is no friction between the axis and its bearing, the reaction acts along a radius, and the sliding motion of the axis on its bearing takes place at right angles to the radius; consequently in this case, too, the reaction neither does work nor has work expended on it. In the actual case, the axis has a sensible radius, and there is friction between it and its bearing; as the friction acts tangentially in a direction opposite to the motion, work must be expended in overcoming it when the axle turns on its bearing.

(*d*) Consider the case of a rod firmly held at one end, and let a force be applied at the other end tending to stretch it. If we suppose the rod absolutely inextensible, no work is expended on the internal forces which hold the parts of the rod together. But if the rod is stretched, the force clearly does work, since its point of application moves forward; and this work is expended in overcoming the resistance which the internal forces offer to the lengthening of the rod. This is only an example of what is universally true, viz. that if a body were perfectly rigid, no work would be expended on the internal forces; but when a body undergoes any change of form or volume, work must be expended on the resistances offered to the change by the internal forces.

Ex. 86.—A body weighing 500 lbs. slides on a rough horizontal plane, the co-efficient of friction being 0·1; how much work must be expended on, or done against, friction while the body slides over 100 ft.? Here the friction is a force of 50 lbs. acting in a direction opposite to the motion; consequently when the motion take place through 100 ft. there must be 50×100, or 5,000 units of work (foot-pounds) expended.

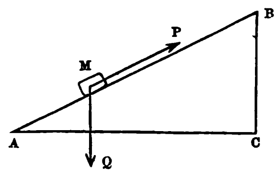
Ex. 87.—The radius of the axle of a fly-wheel is 6 in.; the constant pressure of the axle on its bearing is 5 tons; the coefficient of friction between axle and bearing is 0·075, and 20 turns are made a minute; the whole amount of work expended on the friction in one minute may be found thus:—the friction is a force of $11{,}200 \times 0{\cdot}075$, or 840 lbs. The space through which this resistance is overcome is 20 circumferences of the axle, or 62·8318 ft.; and hence the work expended on friction is $840 \times 62{\cdot}8318$, or 52,779 foot-pounds.

Ex. 88.—A train weighing 100 tons moves over 30 miles along a horizontal road; the resistances are at the rate of 8 lbs. a ton; the whole

quantity of work expended will be $800 \times 30 \times 5280$, or 126,720,000 foot-pounds.

72. *Work of drawing a body up or down an inclined plane.*—Let A B be the inclined plane, B C its vertical height, and A C its horizontal base. The body M, whose weight is Q, is placed upon it, and is supposed to be drawn up the plane by a power P, acting on M in a direction parallel to the plane. We will denote the co-efficient of friction between M and the plane by μ. When M is moved from A to B, work must be expended, (1) on gravity; (2) on friction. It is plain that B C is the distance through which the body is lifted, measured in the direction of gravity (Art. 70); and therefore the work expended on gravity is $Q \times BC$. We have already seen (Ex. 58) that the perpendicular pressure on the plane is $Q \times AC \div AB$, in the case supposed, viz. when the direction of the traction is parallel to the plane. Hence the friction equals $\mu Q \times AC \div AB$, and since friction acts along B A, the work expended on it must be, friction $\times$ A B, or $\mu Q \times AC$. Hence the whole work expended is

FIG. 82.



$$Q \times BC + \mu Q \times AC.$$

This result is of considerable importance, and may be stated thus :—When the direction of traction is parallel to the inclined plane, the work expended is the same as would be expended in lifting the body through the vertical height of the plane ($Q \times BC$), together with what would be expended in drawing the body along an equally rough horizontal path, equal in length to the base of the plane ($\mu Q \times AC$).

If we suppose the body to be drawn down the plane, there will be the same amount of work expended on friction, viz. $\mu Q \times AC$, but in the descent gravity will do

work, viz. $Q \times BC$. Hence the work expended in drawing the body down the plane will be

$$-Q \times BC + \mu Q \times AC.$$

If in this expression the former term is greater than the latter, gravity does more work than what is expended on friction, and the body slides down the plane with a continually increasing velocity.

Ex. 89.—A train weighs 100 tons; resistances are 8 lbs. a ton; how many units of work must be expended in raising it to the top of an incline a mile long of 1 in 70 (in any case of this kind 1 in 70 means 1 vertical to 70 horizontal, and the *horizontal* length of the plane is a mile). Here the work expended on friction is 800 × 5280, or 4,224,000 foot-pounds; the work expended on gravity is 224,000 × 5280 ÷ 70, or 16,896,000 foot-pounds; so that the whole work expended is 21,120,000 foot-pounds.

Ex. 90.—In the last case, if the motion of the train had been down the incline, gravity would have done a quantity of work measured by 16,896,000 foot-pounds, and of these only 4,224,000 would have been expended on friction; consequently an excess of 12,672,000 foot-pounds would remain unexpended, and this would cause a very considerable increase in the velocity of the train. If it were wished that the train should run down the incline without increase of velocity, it would be necessary to increase the resistances, which would be done by putting on the break. In the present case it would be necessary to increase the friction from 8 lbs. a ton to 32 lbs. a ton, if the train is not to acquire any additional velocity when going down the incline.

73. *General relation between forces acting on a machine.*—A machine is a system of pieces so arranged as to enable one force to overcome another force. Thus, a pumping engine is a system of pieces by which the elastic force of steam is applied to lifting water out of a mine. The force which keeps the machine in motion is commonly and conveniently called the *power*, and its point of application the *driving point*; the force, to overcome whose resistance is the object of the machine, is called the *weight*, and its point of application the *working point.* It is scarcely necessary to remark that there may be machines set in motion by more than one power, and

having for their object to overcome the resistance of more than one force; but these cases need not be further specified. In all cases the motion of the machine will give rise to friction, and therefore the power will have to overcome not only the weight, but also the friction exerted between the parts, and it may be other passive resistances of a sensible amount, such as resistance of the air.

When a machine is in motion several cases may arise:—(*a*) The *power* may be just sufficient to overcome the *weight* and other resistances; in this case the motion is uniform. It must be observed, however, that the motion of the machine must have originated when the action of the forces differed in some way from their action when the motion is uniform. While the motion is uniform, the power, the weight, and the other resistances form a system of mutually balanced forces, and it is plain that such a system could not have originated the motion. The motion, however, having been once given will continue uniform under the action of a system of balanced forces. (*b*) The power may not be sufficient to overcome the *weight* and other resistances; in this case the motion continually diminishes, and the machine at length comes to rest or even has the direction of its motion reversed. (*c*) The *power* may be more than sufficient to just overcome the *weight* and other resistances; in this case the motion of the machine is accelerated. It is scarcely necessary to remark that the second and third states can never be of long continuance: if they were allowed to go on, in the one case the machine would cease to move, or move in the opposite direction to that intended; in the other case the motion would become dangerously rapid. In many instances, however, these two states alternate, the motion being accelerated for a short time, and then for a short time retarded.

When the motion is uniform the following relation

will exist between the forces acting on the machine:—*The work done by the power at the driving point equals the work expended on the weight at the working point, together with the work expended on the passive resistances, and this will be true during the whole or any part of the motion.* If we suppose the machine to move from a state of rest, and after a certain time to return to a state of rest, the same relation will hold good with reference to the work done, and the work expended during the *whole time.*

Let the power and the weight be denoted by P and Q, let p denote the space described by the driving point measured in P's direction, and q the space described by the working point measured along the line of Q's action, but of course in a direction exactly opposite to that in which Q acts, so that $\mathrm{P}p$ is the work done by P, and $\mathrm{Q}q$ is the work done against or expended on Q. If we suppose the machine to move uniformly during the given time, and the passive resistances to be zero, so that no work is expended on them, we shall have

$$\mathrm{P}p = \mathrm{Q}q.$$

In the following articles we will apply this principle to determine the relation between the power and the weight in certain simple machines, when in a state of equilibrium. But before doing so, it will be well to notice some of the consequences of this principle. Suppose we know that $\mathrm{P} = \frac{1}{2}\,\mathrm{Q}$, then $p = 2\,q$, i.e. if the machine is such that the power is half the weight, the distance through which the driving point moves is twice that through which the working point moves; similarly if the power is one-third of the weight, the driving point describes three times the distance described by the working point, and so on in any proportion.

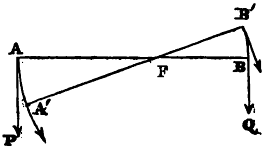
Ex. 91.—A man working with a force of 30 lbs. raises a weight of 18 cwt. through a height of 10 ft. The work done at the working point is $18 \times 112 \times 10$, or 20,160 foot-pounds; if the machine worked entirely with-

out friction, this number and no more must have been done by the man, who consequently must have worked the driving point of the machine (say the handle of the winch which he turned) through a distance of $20{,}160 \div 30$, or 672 ft.

Ex. 92.—A train weighing 50 tons subject to resistances at the rate of 7 lbs. per ton, moves from rest, and after describing 5 miles comes again to rest; the work done against the resistances must be $350 \times 5 \times 5280$, or 9,240,000 foot-pounds. And this is likewise the work done by the steam at the driving point, whether the velocity with which the 5 miles are described is uniform or not,

74. *The lever.*—Let AB be the lever, which we will suppose to be without weight and to turn freely on the fulcrum F. We will suppose the power P and the weight Q to act at A and B respectively at right angles to AB. Now suppose the lever to be turned round F, and to be brought into the position $A'B'$. If we suppose the forces to act at right angles to the rod during the motion, the work done by P will be $P \times AA'$, and that expended on Q will be $Q \times BB'$. The reaction of the fulcrum does no work, and consequently

FIG. 83.



$$P \times AA' = Q \times BB'.$$

Now

$$\frac{AA'}{AF} = \frac{BB'}{BF};$$

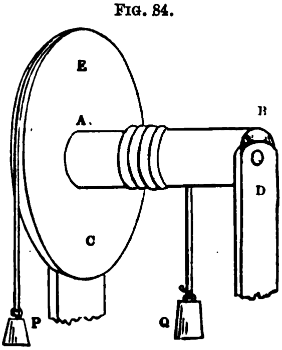
therefore

$$P \times AF = Q \times BF,$$

when P and Q are in equilibrium. The same result, it will be observed, as that obtained by the principle of moments when the machine is at rest.

75. *The wheel and axle.*—The simplest form of this machine is shown in the annexed figure. AB is a cylinder of moderate radius to which is firmly attached a cylinder of larger radius, EC. The former is called the axle, the latter the wheel. An axis of sufficient strength passes lengthwise through the middle of the machine; its ends, one of which is shown at D, rest on bearings, and support the machine. The weight is fastened to one end of a rope, which is coiled round the axle, the other end being fastened to the axle to keep it from slipping. The power may be conceived to be similarly attached to the wheel.

FIG. 84.



It is evident on inspecting the figure that if the machine makes one

turn P will descend through a distance equal to the circumference of the wheel, while Q will ascend through a distance equal to the circumference of the axle. Hence the work done by P equals P × circumference of the wheel, while the work expended on Q equals Q × circumference of the axle; and these will be equal to each other if the motion is uniform, and if the machine turns on its bearing without friction. Now the circumferences of these circles are in the same ratio as their radii.

Therefore P × radius wheel = Q × radius axle.

Instead of the wheel a winch is very commonly used; this is a matter of convenience, and provided P acts always at right angles to the arm by which it turns the machine, the relation between it and Q will be the same as that given above.

76. *The single fixed pulley* is merely a wheel capable of turning on an axis which rests on a fixed support. A groove is cut on the rim of the wheel in which a rope can rest. It is plain that when P falls through any distance, Q rises through the same distance, so that if we suppose the rope to be perfectly flexible and the pulley to turn on the axis without friction, we must have P = Q. The same result can be arrived at by observing that, if we suppose the radii of the wheel and axle to become equal, the relation between the forces must be the same as in the single fixed pulley.

FIG. 85.

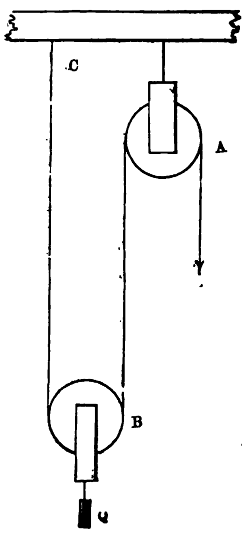


FIG. 86.

77. *Systems of pulleys.*—Pulleys may be combined in many ways, and thus form various systems; of these we will notice two, viz.:—

(*a*) *The single moveable pulley.*—The power P (fig. 85) is fastened to the end of a rope which passes over a fixed pulley A, and under the moveable pulley B, and is then fastened to a beam at the fixed point C. It is supposed that the parts of the rope are all vertical. Now suppose that Q is raised a foot, the parts of the rope between A and B, and between B and C, are each shortened by one foot, and consequently P falls two feet. So that when the work done by P is P × 2, that expended on Q is Q × 1. And therefore

$$2\,P = Q.$$

It will be remarked that Q includes the weight of the moveable pulley.

(*b*) The tackle of four sheaves (fig. 86) consists of two blocks, with two pulleys or sheaves in each, the upper block is fastened to a fixed beam, the lower carries the weight. The rope is fastened to the upper block, and passes in succession under the sheaves of the lower block and over those of the upper block; to the end of the rope the power is applied, as shown in the figure. There are, it will be seen, four parallel parts of the rope besides that to which P is fastened. If we suppose Q to be raised a foot (which will involve an expenditure of $Q \times 1$ units of work), each of the parallel parts of the rope is shortened by one foot, and P will descend 4 feet, thereby doing $P \times 4$ units of work. Hence

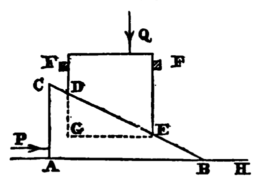
$$4P = Q.$$

It must be remembered that Q includes the weight of the lower block.

78. *Remark.*—The student must bear in mind the suppositions on which we have obtained the results in the last Article, viz. (1) That the ropes are perfectly flexible, (2) that the pulleys turn without friction on their axis Neither supposition is strictly correct. In the case of the single fixed pulley P is greater than Q, but not by very much. When several pulleys are formed into a system, P is greater than Q, and in a proportion which increases very rapidly with the number of pulleys employed. Thus in ordinary cases with the tackle of six sheaves P is found to be a quarter or a third, instead of a sixth, of Q.

79. *The wedge.*—The action of the forces on the wedge is, perhaps, best seen when it takes the form shown in the annexed figure:—ABC is an inclined plane moving on a fixed table AH; DEF is a piece capable of moving up and down between guides F and F, the two pieces touching each other on the inclined surface DE. The power (P) is applied at the back of the wedge AC; the weight (Q) bears on the top of DEF; in all ordinary cases P and Q are so large that the weights of the pieces need not be considered. Draw EG and DG parallel and perpendicular respectively to AH; it is evident that if P causes the wedge to advance through a distance GB, Q will be raised through a height equal to DG. So that the work done by P equals $P \times GB$, and that expended on Q equals $Q \times DG$. Hence

FIG. 87.



$$P \times GB = Q \times DG,$$

if the frictions are neglected. In reality the frictions are very great, but the exact determination of their effect is far from easy: an approximate determination can be made as follows:—Let μ denote the coefficient of friction between the wedge and the table. Now putting out of account the friction of the guides, the perpendicular pressure on the table must be Q, and the friction of the table μQ; consequently when the wedge is moved

forward a distance GE, the work expended on that friction must be $\mu Q \times GE$. It may be assumed without much error that an equal amount of work will be expended on the friction of the surface DE. Hence we have

$$P \times GE = Q \times DG + 2\mu Q \times GE.$$

Ex. 93.—Suppose DG to be unity, and GE to be 10, and μ to equal 0·42, we have $10P = 9{\cdot}4Q$; whereas if the frictions had been neglected, we should have had $10P = Q$. So that, for instance, if Q is 1000 lbs., it will not be moved if P is less than about 940 lbs.; whereas, if the surfaces had been smooth, it would have moved if P had exceeded 100 lbs.

It might be thought at first sight that, since P is so nearly equal to Q, the wedge would be of but little use. This, however, is by no means the case. If the force P is removed, the friction of the surfaces will prevent Q from forcing back the wedge. Now suppose that the wedge, instead of being urged forward by a constant force, is struck by a mallet. We shall see further on that such a blow causes an enormous force to be exerted for a very short time; thus even when Q is large, it is for the instant made to yield, and the wedge advances through a small space; and as Q cannot force the wedge back, a succession of such blows will have the effect of lifting Q through a considerable height.

80. *The screw.*—Suppose ABC to be a right-angled triangle cut out of paper; suppose it to be twisted round so that its base AC becomes the circumference of a circle, as shown in Fig. 88. The hypotenuse of the triangle will take the form of the helix, i. e. of the curve to which the thread of a screw would be reduced if its thickness were exceedingly small. The height AB of the triangle would be the distance measured parallel to the axis between two consecutive turns of the thread, or, what is called the *pitch* of the screw; and the angle BCA is the inclination of the thread. It is evident that there are two ways of twisting the triangle ABC, viz. according as one face of the triangle or the other is turned inward, the one produces what is called a right-handed, the other a left-handed, helix. Fig. 88 shows a right-handed helix. The relation between this curve and an ordinary square threaded screw will be evident on an inspection of the accompanying figure (89), which represents a right-handed screw. It will be noticed that the thread next to the observer ascends to his right hand. *The screw works in a* nut, on the inside of which a groove is cut, which

FIG. 88.

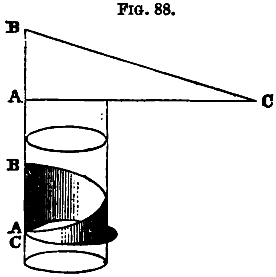
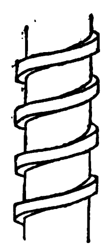


FIG. 89.

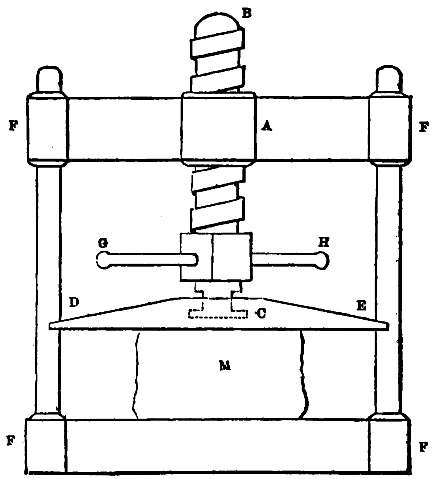


the thread exactly fits; this groove is called the companion screw. If the nut is fixed, the end of the screw will advance or recede through a distance equal to the pitch when the screw is turned once. But if the screw can neither advance nor recede, the nut, being held between guides, will either recede or advance when the screw is turned.

In order to study the relation between the power and the weight in the screw, we will take the case of the Screw Press, which may be briefly described as follows:—

F F F F is a strong frame; at A in the middle of the cross-piece is the nut in which the screw B C works; the end, C, of the screw is attached loosely to the piece D E, so that the screw is free to turn, while D E is constrained to move up and down between the guides. When the arm G H is turned in the proper direction, the screw advances, and the substance M is compressed between D E and the fixed base of the frame. In this case the *weight* is the reaction of the mass M which is undergoing compression.

FIG. 90.



It is obvious that when the end of the arm, on which the power acts, describes a complete circle, the weight will be moved through a distance equal to the pitch of the screw. Let P and Q denote the power and the weight, a the distance from the axis of the screw at which the power acts, and h the pitch of the screw. Now when Q is moved through a distance h, P's point of application moves through a distance $2\pi a$, i.e. the circumference of a circle whose radius is a. Hence the work done by P is $2\pi a$ P, while the work expended on Q is Q h; therefore, if all the frictions were neglected, we should have

$$2\pi a P = Q h. \qquad (1)$$

In reality, however, the frictions greatly modify the result. A sufficiently close approximation to the relation between P and Q when friction is taken into account, can be obtained as follows:—The principal frictions are (1) that between the thread of the screw and the companion, (2) that between the end C and its surface of contact with D E; the whole perpendicular

pressure in both cases being nearly or exactly equal to Q. Let r denote the radius of the screw and μ the coefficient of friction between the screw and companion; then the work expended on the first friction in one turn of the screw will equal $\mu \text{Q} \times 2\pi r$. Let ρ denote the radius of the end and μ' the coefficient of friction between the end of the screw and D E; then, assuming that the pressure is uniformly distributed, it admits of proof that the work expended on the second friction in one turn of the screw equals $\frac{2}{3}\mu' \text{Q} \times 2\pi\rho$. Hence we obtain the following relation, which is, however, only approximately true—

$$2\pi a \text{P} = \text{Q} h + 2\pi r \mu \text{Q} + \tfrac{4}{3}\pi\rho\mu'\text{Q}. \qquad (2)$$

Ex. 94.—If the arm of the screw is 3 ft. long, the pitch 2 in., the radius of the screw and of its end 3 in., and both μ and μ' equal to 0·18, we shall have

$$10{,}000\, \text{P} = 338\, \text{Q};$$

but if we had neglected the frictions, we should have had

$$10{,}000\, \text{P} = 88\, \text{Q}.$$

In other words, the force required to produce a given compression (Q) is nearly four times as great as would be required if there were no friction; e.g. if it were required to compress the body with a force of 10,000 lbs., P must equal 338 lbs., whereas 88 lbs. would be enough if there were no friction.

81. *The modulus of a machine.*—We have already seen (Art. 73) that the work done by the power is expended partly on the passive resistances, and partly in overcoming the resistance of the weight, i.e. in doing work at the working points of the machine. The former expenditure may be unavoidable, but is otherwise useless, and any practicable reduction of its amount would be so much gain; the latter work is expended usefully, i.e. it is employed in doing the work which it is the object of the machine to accomplish. Let U denote the work done in a certain time by the power, and U_1 the work expended usefully in the same time; then, assuming the machine to move uniformly, U_1 will be less than U, and we may express the relation between them by the equation

$$\text{U}_1 = \text{K}\, \text{U},$$

where K stands for a proper fraction. Now, in most

machines, so long as they are in a state of uniform motion, K will be constant, and is called the modulus of the machine. Thus, the work done by the power in the case of a water-wheel is that done by the stream of water, in descending from the mill-race to the tail-race; it has been found that in wheels of the best construction only about 7-10ths of this work is usefully employed, so that in this case

$$U_1 = 0{\cdot}7\ U,$$

and the modulus of such a machine is 0·7. In other words, for every 10 units of work done by the falling water only 7 are usefully employed. It may be added that 0·7 is a large modulus; in water-wheels of various constructions the modulus has been found to vary from 0·25 to 0·75.

In the case of machines in which the relation of the power and the weight is known the modulus can be easily inferred.

Ex. 95.—The circumference described by the handle of a winch is 8 ft.; a man works it with a constant force of 30 lbs. in a tangential direction; in 50 turns he is observed to raise a weight of 8 cwt. through a height of 10 ft. What is the modulus of the machine?

Here, the work done at the driving point (U) is $30 \times 8 \times 50$, or 12,000 foot-pounds; while the work done at the working point (U_1) is 896×10, or 8960 foot-pounds. Consequently $U_1 = \frac{8960}{12{,}000}\ U$, or $U_1 = 0{\cdot}747\ U$, i.e. the modulus of the machine is about 0·75.

Ex. 96.—Determine the modulus of the screw press in Ex. 94.

The work done by P in one turn of the screw is $72\,\pi\,P$, or $226{\cdot}19\ P$, while the work expended on Q is $2\ Q$. Therefore

$$\frac{U_1}{U} = \frac{2\,Q}{226{\cdot}19\,P}.$$

But we know that $338\,Q = 10{,}000\,P$,

therefore $$\frac{U_1}{U} = \frac{2}{226{\cdot}19} \times \frac{10{,}000}{338} = 0{\cdot}26,$$

i.e. the modulus of the machine is 0·26. In other words, for every 100 units of work done by P, only 26 are expended usefully; the remainder is employed in *overcoming* the friction of the parts of the machine.

82. *Comparison of the efficiency of agents.*—If it is required to compare the efficiency or working power of two agents, we have only to compare the number of units of work done by them in a given time. Suppose that two agents do 10,000 foot-pounds per minute, their working powers for the time are equal. If they acted on machines duly adapted to them, they would overcome equal resistances through equal spaces in a minute. Thus, putting the friction of the machines out of the question, if they were required to raise water to a height of 50 ft., each would raise 20 gallons or 200 lbs. per minute. If, however, the one does 10,000 and the other 20,000 foot-pounds per minute, the efficiency of the second agent is double that of the first, e. g. upon the same supposition the second agent would raise 40 gallons while the first was raising 20 gallons. Similar reasoning can be applied in other cases, and thus it follows that the efficiency of an agent can be measured by the number of units of work done per minute while he is working. This number, however, in the case of some agents may be very large, and accordingly it is found convenient to use a superior unit called a *horse-power*, which may be thus defined:—

An agent doing 33,000 foot-pounds per minute works with one horse-power.[1]

Several practical questions regarding the power of the agent needed to perform a certain work can be solved by the principles we have now considered. Whatever work is done at the working point must be done by the agent, and, besides this, work will have to be expended on the friction and other prejudicial resistances of the parts of the machine. An equation can be formed between the number of units of work done by the agent at the driving

[1] The student must not suppose that this is the working power of an average horse. A horse-power is simply a word with the meaning above *assigned to it.*

point, and the number that must be expended at the working points of the machine; the resistances being taken into account, if necessary, by introducing a modulus. One of the quantities introduced into this equation will be the unknown quantity required, and the equation will give the means of determining it. Two examples will sufficiently illustrate the method of solving these questions.

Ex. 97.—A steam-engine works under an average pressure of 15 lbs. per square inch on a piston with an area of 2000 sq. in.; it makes 8 strokes per minute, and each stroke is 10 ft. long. Assuming the modulus of the machine to be 0·55, how many gallons of water will it raise in one hour from a depth of 200 ft.

Here the force which does the work is 30,000 lbs., and in one hour it acts through a distance of $10 \times 8 \times 60$ ft.; therefore the work done by the agent in one hour is $30{,}000 \times 10 \times 8 \times 60$ foot-pounds, and of this $0{\cdot}55 \times 30{,}000 \times 10 \times 8 \times 60$ foot-pounds are usefully employed. But if x is the required number of gallons, a weight of $10x$ lbs. must be raised through 200 ft. in one hour. Hence

$$10x \times 200 = 0{\cdot}55 \times 30{,}000 \times 10 \times 8 \times 60,$$

or $$x = 39{,}600 \text{ (gallons raised per hour).}$$

Ex. 98.—The section of a stream is 6 sq. ft., and its velocity through the section 264 ft. per minute; there is a fall of 20 ft. by which the stream turns a wheel with a modulus 0·64. Assuming that each horse-power grinds a bushel of corn per hour, find how much corn the wheel grinds in an hour.

The weight of water that falls per minute is $6 \times 264 \times \dfrac{1000}{16}$ lbs., and hence the units of work usefully employed per minute are $0{\cdot}64 \times 20 \times 6 \times 264 \times 1000 \div 16$ foot-pounds, and therefore the horse-power of the wheel will be this number divided by 33,000 or 38·4. Consequently the wheel will grind 38·4 bushels per hour.

83. *Animate agents*.—When we consider the work done by animate agents, we find peculiarities which will be best explained by considering a particular case. Suppose a man of average strength employed to turn a winch, it is found that he will be able to do about 1,300,000 foot-pounds a day. That is to say, the fatigue experienced in doing this work will be such that it could be repeated day

after day without injury to his health. If he did much more than this for several days successively he would break down. He could, however, do these 1,300,000 foot-pounds daily, only on condition (*a*) of exerting at right angles to the end of the winch a force of about 18 lbs.; (*b*) of causing the end of the winch to move at the rate of about 150 ft. per minute; and (*c*) of working for eight hours a day. If he exerted a much greater force, or worked at a much greater speed, he would not be able to work day after day for eight hours, and it would be found that, if the length of the day's work were reduced to meet the altered circumstances, he would not do as many as 1,300,000 foot-pounds a day. This, of course, is true of a man of average strength, and would require modification to adapt it to a man above the average strength.[1] Now what is true when a man is set to turn a winch is true, *mutatis mutandis*, of other ways of employing human labour, and the labour of animals. In all cases there will be a certain speed, a certain force, and a certain length of day's work with which the agent will do the greatest amount of daily work, i.e. work that can be repeated day after day without injury to his health. It is, of course, difficult to assign exact numbers, because in particular cases an agent's working power may greatly exceed or fall below the average proper to his class. Subject, however, to this reservation it may be stated that:—(1) A man can raise his own weight (145 lbs.) vertically, by going up a ladder or walking along a gentle incline, at the rate of 29 feet a minute, and can continue doing this for eight hours, thereby doing rather more than 2,000,000 foot-pounds a day; (2) a man can work on a capstan for eight hours a

[1] The following case was actually observed:—A man, doubtless above the average strength, worked day after day for three months on a winch; on an average he exerted a force of 31·5 lbs. with a velocity of 100 ft. per minute, for 7 hours a day, thereby doing about 1,323,000 foot-pounds a day (Poncelet, *Méc. Indust.* p. 236).

day, exerting a force of 27 lbs., at the rate 116 ft. per minute, doing thereby about 1,500,000 foot-pounds a day; (3) that a draught horse yoked to a cart, and walking, can work for ten hours a day, exerting a force of 154 lbs. at the rate of 177 ft. per minute (or two miles per hour), and doing about 16,500,000 foot-pounds a day.[1] What limits the amount of daily useful work is the degree of fatigue which the agent undergoes in its performance. If the degree of fatigue is excessive, the work cannot be continued. It may be further observed that the average degree of fatigue can be incurred by modes of working which largely reduce the daily amount of useful work. Thus, when a labourer is employed to raise earth with a shovel and throw it into a truck, he can lift 20 tons weight a day to a height of 6 ft., i.e. his useful work is only 268,800 foot-pounds; some men, however, are so strong as to do a seventh more than this, i.e. 307,200 units a day. This, it must be borne in mind, is labour which exceeds in severity almost any other description of work; yet it will be seen that the amount of useful work is little more than one-fifth of that done by the labourer mentioned in Note, p. 108.[2] It is pretty plain that the fatigue is

[1] These numbers are reduced from those given by Poncelet, *Méc. Indust.* p. 235. The first and second estimates are adopted without material change by Mr. Rankine (*Applied Mechanics*, p. 626). In the third case, that of the draught horse, Mr. Rankine estimates the day's length at 8 hrs., the force exerted at 120 lbs., the speed at 216 ft. a minute (about 2½ miles an hour), and the daily useful work at 12,441,600 foot-pounds. The estimate given by Weisbach ('Lehrbuch der Mech.' vol. ii. p. 289) on the authority of Gerstner is 120 lbs. force exerted at the rate of 240 ft. per second for 8 hrs. a day. The foot and pound, however, are slightly greater than the English foot and pound, and the result when reduced to English units gives about 14,750,000 foot-pounds a day; a result just midway between Poncelet's and Rankine's estimates.

[2] A full day's work done by an English navvy 'consists of fourteen sets a day. A "set" is a number of wagons, in fact a train. There are two men to a wagon. If the wagon goes out fourteen times, each man has to fill seven wagons in the course of the day. Each wagon contains two and

mainly incurred in raising the body, arms and spade in the act of throwing the earth.

Ex. 99.—Suppose a weight of 12,000 lbs. has to be raised by one man working on a screw-press; and suppose that it is so arranged that he can work on it in the same manner as a man can work on a capstan. As the man can do 1,500,000 foot-pounds a day, we may conclude at once that, if we could neglect friction, he would lift the weight to a height of 125 ft. The machine, however, must be adapted to enable him to lift this weight by the exertion of a force of 27 lbs.; if, then, he works on an arm 8 ft. long, the pitch of the screw (Art. 80) must be $2\pi \times 96 \times 27 \div 12{,}000$, or $1\frac{1}{3}$ in. (very nearly). The introduction of this machine, however, necessitates that part of the work done by the man should be expended on friction. We therefore ought to solve the following example.

Ex. 100.—In the last example suppose the pitch of the screw to be 2 in.; its diameter and that of its end 6 in.; coefficient of friction 0·1. Find (1) what must be the length of the arm; (2) through what height the man will raise the weight in the course of a day.

If we denote the length of the arm in inches by x, we obtain from Art. 80, eq. (2)

$$2\pi x \times 27 = 12{,}000 \times 2 + 2\pi \times 3 \times 0{\cdot}1 \times 12{,}000 + \tfrac{4}{3}\pi \times 3 \times 0{\cdot}1 \times 12{,}000;$$

whence $$x = 364 \text{ in.};$$

so that in one turn of the screw P does 5142 foot-pounds; as, in fact, can be obtained at once from the above equation. Hence in the course of the day the man can turn the screw $1{,}500{,}000 \div 5142$ (say 300) times. He can therefore raise the weight through only 300×2 in., or 50 ft. The student will observe that for the man to raise the weight at all he must use the machine, but this advantage he purchases by the loss of nearly $\frac{2}{3}$rds of his working power, which are expended on the friction of the machine.

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a quarter cubic yards. The result is that each man has to lift nearly twenty tons weight of earth on a shovel over his head into a wagon. The height of the lifting is about six feet. This is taking it at fourteen sets a day; but the navvies sometimes contrive to get through sixteen sets, and there are some men who will accomplish that astonishing quantity of work by three or four o'clock in the afternoon—a result, I believe, which is not nearly equalled by the workmen of any other country in the world' (Mr. Ballard's evidence, quoted in Sir A. Helps's *Life of Mr. Brassey*, p. 77). This corresponds very closely with Poncelet's estimate, who considers that a man raising earth with a spade is capable of doing about 280,000 units of useful work per diem, if he raises the earth at each throw $5\frac{1}{4}$ ft. in loads of about 6 lbs. each, and works for 10 hrs. a day (*Mécanique Indust.* p. 234).

QUESTIONS.

1. State what is meant by the work done by a force, and by the work expended on, or done against, a resistance.

2. Define a unit of work. When a unit of work is called a 'foot-pound,' what is implied as to the units of force and distance?

3. A weight of 800 lbs. falls from a height of 12 ft.; what work has been done by gravity? *Ans.* 9600 ft.-pds.

4. A stream of water has a section of 6 sq. ft.; the water moves through the section at the rate of 3 miles an hour. If the water falls down 20 ft., how much work is done by gravity per hour? *Ans.* 118,800,000 ft.-pds.

5. When a system of weights is lifted, different weights being raised different heights, what is the whole amount of work done?

6. Three weights of 4, 5 and 6 lbs. are at the same point; they are raised vertically through 5, 2 and 10 ft. respectively; show that their centre of gravity is raised 6 ft. vertically, and verify in this case the principle stated in Art. 69.

7. A rope 500 ft. long weighing 2 lbs. a ft. hangs by one end; how much work must be expended in winding it up? *Ans.* 250,000 ft.-pds.

8. Mention some circumstances in which a force does no work, and illustrate your answer by reference to the motion :—(*a*) of a point sliding on a horizontal plane; (*b*) of a wheel; (*c*) of a body turning on an axle; and (*d*) by the case of a rod under the action of a stretching force.

9. A train weighing 60 tons moves along a horizontal road, the resistances being at the rate of 10 lbs. a ton; how much work must be expended in drawing the train the distance of one mile? *Ans.* 3,168,000 ft.-pds.

10. The weight sustained by one wheel of a carriage is 500 lbs.; if the resistance of the road to the turning of the wheel is equivalent to lifting the wheel 1 in. in each turn, what work must be expended on the resistance in the course of a mile, the circumference of the wheel being 5 yds.? *Ans.* 14,667 ft.-pds.

11. In the last question suppose the circumference of the axle to be 8 in., and the friction between it and its bearing to be 40 lbs.; how much work is expended on the friction of the axle in a run of one mile? *Ans.* 9387 ft.-pds.

12. Determine the number of units of work needed to draw a body up or down an inclined plane; the direction of traction being parallel to the plane.

13. The base and height of an inclined plane are 40 ft. and 30 ft. respectively; a body weighing 1 ton is drawn up the plane by a force acting

parallel to the plane; the coefficient of friction between body and plane is $\frac{1}{8}$; how much work is expended? *Ans.* 97,067 ft.-pds.

14. A train weighing 50 tons runs down an incline 2 miles long of 1 in 100; how much work is done by gravity? If this is wholly expended in overcoming friction, what must the friction be in lbs. per ton?

Ans. (1) 11,827,200 ft.-pds.; (2) 22·4.

15. Define a machine, power, weight, driving point, working point. State briefly the circumstances under which the motion of the machine is uniform, retarded or accelerated.

16. When the motion of a machine is uniform, what relation must exist between the power, the weight, and the passive resistances?

17. A machine is so contrived that a weight of 1 ton can be just moved by a force of 28 lbs.—putting passive resistances out of the question—what conclusion can we draw as to the distances described by the working point and driving point of the machine?

18. In Art. 72, if the body moves up the plane with a uniform velocity under the action of a constant traction (P), show that the following relation holds good—

$$P.AB = \mu Q.AC + Q.BC.$$

19. A smooth plane has a base 12 ft. and a height 5 ft.; a weight of 572 lbs. is supported on it by a force (P) acting parallel to the plane; show by the last question that P equals 220 lbs. Verify this result by a process similar to that used in Ex. 52.

20. Find, by the principle laid down in Art. 73 the relation between the power and the weight in a lever of the first order; and apply similar reasoning to obtain the relation in the case of a lever of the second or third order.

21. Find the relation between the power and the weight in the wheel and axle.

22. Find the relation between the power and the weight (*a*) in a single fixed pulley, (*b*) in a single moveable pulley, (*c*) in a tackle of four sheaves. Mention in general terms what is the modification produced in these relations by friction.

23. In two blocks of three sheaves each, it is found that a power of 390 lbs. is required to raise a weight of 1000 lbs.; when the weight is raised through 10 ft., how much work is done against the passive resistances? *Ans.* 13,400 ft.-pds.

24. Find the relation between the power and weight in the wedge. Explain an approximate method of taking account of friction.

25. Adapt the case of the wedge (Art. 79) given in the text to that in which the plane A B is not at right angles to Q's direction, and show that

P bears to Q the same ratio that the back bears to the length of the wedge.

26. The back and length of a wedge are 3 in. and 18 in. respectively; the surfaces in contact are metal on metal; given that Q is 1000 lbs.; find P, first neglecting friction, secondly taking account of friction.

Ans. (1) 167 lbs.; (2) 527 lbs.

27. What is the advantage that is actually gained by the use of the wedge?

28. A piece of metal is held, merely by friction, between a pair of pincers; from jaws to rivet is 2 in., from rivet to the grasp of the hand is 18 in.; the force exerted on each handle is 100 lbs.; what force would be required to pull the metal from between the jaws? *Ans.* 324 lbs.

29. Describe briefly the form of a helix; and distinguish between a right-handed and left-handed screw.

30. The diameter of a screw is 3 in., the thread makes one turn per inch; what is the length of the thread in 1 ft. of the length of the screw, and what is the angle of inclination of thread?

Ans. (1) 113·7 in.; (2) 6°.

31. Describe briefly the screw-press, and find the relation between the power and the weight on the supposition that frictions are neglected. Mention the principal frictions, and give an approximate method of taking account of them.

32. The pitch of a screw is 1 in.; by means of it a weight of 10,000 lbs. has to be raised by a force of 30 lbs.; what should the length of the arm be if there were no frictions? *Ans.* 53 in.

33. In the last question, if we suppose the end of screw to have a diameter equal to that of the screw, and both to be 3 in., and the coefficient of friction to be 0·1, what must be the length of the arm? If the length of arm were unchanged from that determined in the last question, what must the force be? *Ans.* (1) 136·4 in.; (2) 77·1 lbs.

34. What is the modulus of a machine?

35. What is the modulus in Question 33? *Ans.* 0·4.

36. In Question 4, if the stream turns a wheel with a modulus 0·65, how much water would the wheel raise from the bottom of the fall to a height of 100 ft. in one hour? How much would it raise from the top of the fall to the same point? *Ans.* (1) 12,355 cub. ft.; (2) 13,285 cub. ft.

37. How can the working power of two agents be compared?

38. An agent (A) exerts a force of 400 lbs. through 300 ft. in 3 minutes; another agent (B) exerts a force of 80 lbs. through 50 ft. in a quarter of a minute; what is the ratio of A's working power to B's?

Ans. 5 : 2.

39. What is meant by a horse-power? What is the horse-power of the steam in Question 4? If the engine in Ex. 82 makes 11 strokes a minute, what is its horse-power? *Ans.* (1) 60 h.-p.; (2) 160 h.-p.

40. State the leading peculiarities of the working power of men and animals.

41. The barrel of a capstan is 1 ft. in radius, and men work on the handspikes at an average distance of 10 ft. from its axis; how many men are required to raise a weight of 5 tons by this machine? Through what height would they lift it in 2 hours?

Ans. (1) 42 men; (2) 1406 ft.

CHAPTER V.

RECTILINEAR MOTION.

84. *Units of distance and time.*—In all questions of dynamics the units of distance and time are assumed to be *feet* and *seconds* unless the contrary is specified; and in nearly all cases it will be found advantageous to reduce distances and times to feet and seconds if they are given in other units. The primary units from which the foot and second are derived are the standard yard and the mean solar day. The foot is, of course, the third part of the standard yard. What is meant by a mean solar day requires a word of explanation. Suppose the sun to be observed when exactly due south on any given day, and to be observed again when exactly due south on the next day, the interval between the observations is an *apparent* solar day; it is found that apparent solar days are not of exactly equal lengths, and therefore to obtain a unit it is necessary to take the average length of a very large number of such days. The day of average length thus determined is called a *mean solar day*, and a *second* or more, exactly a second of mean solar time, is the 1–86,400th part of a mean solar day. If a common clock were to go perfectly, its 24 hours would make up a mean solar day.[1]

85. *Uniform velocity.*—The velocity of a point is the rate of its motion. If the velocity is uniform, the point describes equal distances in equal times; and consequently

[1] With a sun-dial or a sidereal clock the case would be different.

uniform velocity is measured by the distance described in a unit of time—commonly by the number of feet described in a second. Since in each unit of time equal distances are described, the distance described in any time will be the velocity multiplied by the number of units of time, i.e. if v, t and s denote velocity, time and distance described, we must have

$$s = v\,t$$

if the velocity is uniform. A velocity expressed in one set of units is easily expressed in other units, e.g. a velocity of 30 miles an hour is the same thing as a velocity of 30×5280 ft. an hour, and therefore as $30 \times 5280 \div 3600$ or 44 ft. a second. In the following pages if a velocity is merely expressed by a number, it will always mean feet per second, e.g. a velocity 8 will mean a velocity of 8 ft. per second.

86. *Variable velocity* is measured at each instant by the distance which the body would describe in a unit of time, if it continued to move uniformly from that instant; e.g. we say that a train is moving at the rate of 40 miles an hour, meaning, that if it continued to move for an hour without changing its velocity, it would move over 40 miles. In the same manner if we say that a falling body has at any instant a velocity of 32 ft. a second, we mean to state that, if it continued to move uniformly from that instant, it would describe 32 ft. in a second.

87. *Uniformly accelerated velocity.*—When the velocity of a body is increased by equal amounts in any equal times, its motion is said to be uniformly accelerated, e.g. if a body is moving at a certain instant with a velocity of 8 ft. per second, and at the end of the 1st, 2nd, 3rd . . . . second its velocity is found to be 11, 14, 17 . . . . ft. per second, this would be consistent with its velocity having *undergone* a uniform acceleration in each second of 3 ft.

per second. If the initial velocity is v ft. per second, and if in each second the constant increase is a velocity of f ft. per second, the velocities at the end of the 1st, 2nd, 3rd . . . . second will be $\mathrm{v}+f$, $\mathrm{v}+2f$, $\mathrm{v}+3f$ . . . . and plainly if v is the velocity at the end of t seconds, we shall have

$$v=\mathrm{v}+ft.$$

If the body has no initial velocity, we shall of course have

$$v=ft.$$

If the velocity, instead of being uniformly accelerated, is uniformly retarded, i.e. if it is diminished by equal amounts in equal times, we shall have

$$v=\mathrm{v}-ft.$$

The two points following should be carefully borne in mind: (*a*) When a uniform acceleration is said to have a numerical value, e.g. 5, this means that in each second the velocity is increased by a velocity of 5 ft. a second. (*b*) When the acceleration of a body's motion is said to cease, this of course means that the velocity continues uniform from the instant that the acceleration ceased, e.g. if a body moved from rest under a uniform acceleration 10, its velocity at the end of the fourth second would be 40 ft. a second, and if at that instant the acceleration ceased, it would continue to move uniformly at the rate of 40 ft. a second.

Ex. 101.—If the uniform acceleration is 5 in feet and seconds, a body will have a velocity of 50 ft. a second at the end of 10 seconds, provided it moves from rest. But if at any instant it is moving with a velocity of 40 ft. a second, and the velocity undergoes a uniform acceleration of 5 in feet and seconds, its velocity will be 40 + 50, or 90 ft. a second at the end of 10 seconds. If it had undergone a uniform retardation of 5 in feet and seconds, its velocity at the end of 5 sec. would be 40 − 25, or 15 ft. a second, and if the retardation were continued for 8 sec., the velocity would be reduced to 40 − 40, or zero.

88. *Space described by a body moving for a given time with a uniformly accelerated velocity.*—In order to

make this somewhat difficult determination, two points must be attended to, and the student should carefully make out each separately:—

(*a*) Suppose a body (A) to begin to move with a velocity 8, which is subject to a uniform acceleration 3, so that at the end of the 1st, 2nd, 3rd, 4th second its successive velocities are 11, 14, 17, 20. Suppose a second body (B) to begin to move with a velocity 20, which is subject to a uniform retardation 3, so that at the end of the 1st, 2nd, 3rd, 4th second its velocities are 17, 14, 11, 8. It is plain that in the four seconds A and B describe equal distances; the successive velocities[1] of B being just the same as those of A, only in the reverse order. And this is generally true:—If A's velocity is at first v, and is uniformly increased in t seconds to u, while B's is at first u, and is uniformly diminished in t seconds to v, A and B will describe equal distances in the t seconds.

(*b*) Suppose A and B to move from the same point in opposite directions along the same line, the former with a velocity 7, the latter with a velocity 5; it is plain that they will separate with a velocity 12: at the end of one second they will be 12 ft. apart, at the end of two seconds they will be 24 ft. apart, and so on. Now this will be true whether the separate velocities are constant or not, provided their *sum* is always 12, e.g. if the former velocity were increased to $8\frac{1}{2}$, while the latter was diminished to $3\frac{1}{2}$, they would still be separating with a velocity 12; and this is generally true, so that if the velocities of the bodies were u and v, they are separating with a velocity $u+v$, and provided $u+v$ is unchanged, the bodies will separate with a constant velocity, though their respective velocities are varied.

[1] This is true of all the velocities, e.g. if the velocities at the end of each half-second were considered, they would be in the case of A, 8, $9\frac{1}{2}$, 11, $12\frac{1}{2}$, 14, $15\frac{1}{2}$, 17, $18\frac{1}{2}$, 20, and in the case of B, 20, $18\frac{1}{2}$, 17, $15\frac{1}{2}$, 14, $12\frac{1}{2}$, 11, $9\frac{1}{2}$, 8, and the same is true for all equal intervals, however short.

Let us now apply these principles to the following case:—Suppose a body (A) to begin to move with a velocity V, which undergoes a uniform acceleration f, such that at the end of t seconds its velocity is v. And suppose a second body (B) to begin to move with a velocity v, which undergoes a uniform retardation f, so that at the end of t seconds its velocity is V. It follows from (a) that in t seconds A and B describe equal distances. Moreover, if we suppose them to move in opposite directions from the same point, they will at the end of t seconds be $(V+v)t$ feet apart. For the sum of their velocities is always $V+v$. Thus at the end of three seconds the velocities are $V+3f$ and $v-3f$, the sum of which is $V+v$. Hence half this distance is described by A and the other half by B. Now v equals $V+ft$, and therefore if s denotes the distance described by A, we have

$$s=\tfrac{1}{2}(V+V+ft)\,t,$$

or

$$s=Vt+\tfrac{1}{2}ft^2.$$

It is evident from this that, if the body moves from rest (so that V equals zero), we shall have

$$s=\tfrac{1}{2}ft^2;$$

and that if the body's motion undergoes uniform retardation, we shall have

$$s=Vt-\tfrac{1}{2}ft^2.$$

89. *Relation between velocity acquired and space described in the case of uniformly accelerated motion.*—We have already seen when a body moves from rest and its velocity undergoes a uniform acceleration f, its velocity v acquired at the end of t seconds is

$$v=ft; \qquad (1)$$

and if s is the space described in the same time,

$$s=\tfrac{1}{2}ft^2. \qquad (2)$$

Now from (1) $v^2=f^2t^2$, and from (2) $2fs=f^2t^2$;

therefore $$v^2=2fs \tag{3}$$

If we suppose the body to have an initial velocity v, we can take the two equations

$$v=\text{v}+ft \tag{4}$$

and $$s=\text{v}\,t+\tfrac{1}{2}ft^2, \tag{5}$$

and deduce $$v^2=\text{v}^2+2fs. \tag{6}$$

Similarly in the case of uniformly retarded motion we must use the equations

$$v=\text{v}-ft \tag{7}$$

and $$s=\text{v}\,t-\tfrac{1}{2}ft^2, \tag{8}$$

and we can deduce $$v^2=\text{v}^2-2fs. \tag{9}$$

The nine equations brought together in the present article are of great importance. The student should make himself thoroughly familiar with them. He will observe that they fall into three groups of three equations each: if the first group is known, the other two are easily remembered.

Ex. 102.—A body describes a straight line, and its velocity undergoes a uniform acceleration of 5 in feet and seconds. If it begins to move from a state of rest, the space it will describe in 6 sec. will be $\frac{1}{2}\times 5\times 6^2$, or 90 ft., and at the end of the 6 sec. it will be moving at the rate of 30 ft. a second. If it is asked, how far it will move in the next 6 sec., we may proceed thus:—in 12 sec. the distance described is $\frac{1}{2}\times 5\times 12^2$, or 360 ft.; so that in the second period of 6 sec. it must describe $360-90$, or 270 ft. In a third period of 6 sec. (i.e. from the end of the 12th to the end of the 18th sec. of its motion) it will describe 450 ft. The student will observe that the distances 90 ft., 270 ft., 450 ft., are in the same proportion as the numbers 1, 3, 5.

Ex. 103.—A body moves along a straight line, its velocity undergoes a uniform retardation of 8 in feet and seconds. If at any instant it is moving at the rate of 60 ft. a second, how far will it move before coming to rest, and for how long a time?

We may use Equation (9), and say that if v is the velocity, when s ft. have been described, we must have

$$v^2=(60)^2-16\,s.$$

Now when the body comes to rest, its velocity is zero; if, therefore, we take s to mean the distance that has been described when the body comes to rest, we must have

$$0=(60)^2-16\,s.$$

Therefore $$s=225 \text{ ft.}$$

To find the time, we can reason similarly on Eq. (7), and if t denotes the number of seconds at the end of which the body comes to rest, we must have

$$0=60-8\,t,$$

or $$t=7\tfrac{1}{2} \text{ sec.}$$

So that the body will move for $7\frac{1}{2}$ sec. before coming to rest, and describe 225 ft. We might obtain the answer to the question by a slightly different process, thus:—By means of Eq. (7) ascertain that the motion lasts for $7\frac{1}{2}$ sec.; then, Eq. (8) will give the distance described, viz. $60\times7\frac{1}{2}-\frac{1}{2}\times8\times(7\frac{1}{2})^2=450-225$, or 225 ft.

Ex. 104.—A body moving in a straight line is observed to have described 63 ft. at the end of 3 sec., 162 ft. at the end of 6 sec., and 297 at the end of 9 sec. Was this consistent with a uniform acceleration of its motion, and if so, what was the initial velocity and the amount of the uniform acceleration?

We may reason thus:—If these numbers are consistent with uniform acceleration, let f be its amount, and v the initial velocity; consequently by Eq. (5) the space described in 3 sec. will be $3\,\text{v}+\frac{1}{2}f\times9$, and that described in 6 sec. will be $6\,\text{v}+\frac{1}{2}f\times36$; so that

$$3\,\text{v}+\tfrac{1}{2}f\times9=63,$$
$$6\,\text{v}+\tfrac{1}{2}f\times36=162,$$

or
$$6\,\text{v}+9f=126,$$
$$6\,\text{v}+18f=162.$$

Therefore $$9f=36,$$

or $$f=4,$$

and $$\text{v}=15.$$

Now if $\text{v}=15$ and $f=4$, the space described in 9 sec. would be $15\times9+\frac{1}{2}\times4\times9^2=135+162=297$. So that the spaces described are consistent with uniform acceleration, and the body must have had an initial velocity of 15 ft. a second, and in each second its velocity was increased by a velocity of 4 ft. a second, i.e. its velocity was 19 ft. a second at the end of the first second, 23 ft. a second at the end of the second second, 27 ft. a second at the end of the third second, and so on. The consistency of the distances with uniform acceleration might have been inferred thus:—the distances described in three successive intervals of three seconds each were 63, 162 − 63,

and $297-162$, i.e. 63, 99, 135; as these are in arithmetic progression, the motion may have been uniformly accelerated.

90. When a uniform acceleration is estimated in one set of units, it is easy to estimate it in other units if we bear in mind what is meant by a uniform acceleration. Thus, suppose an acceleration is 20 when estimated in feet and seconds, and we wish to find what it will be if estimated in yards and minutes. We know that in each *second* the body's velocity is increased by 20 ft. a second, i.e. by 20×60 or 1200 ft. a minute, or by 400 yds. a minute, and therefore in each *minute* its velocity is increased by 400×60, or 24,000 yds. a minute. And this is the value of the acceleration required. In other words, to say that in each *second* a body acquires an additional velocity of 20 *ft.* a *second*, is to say that in each *minute* it acquires an additional velocity of 24,000 *yds.* a *minute.*

Ex. 105.—A body moves along a straight line, and its velocity undergoes a uniform acceleration of $3\frac{2}{3}$, reckoned in inches and seconds; express this acceleration in miles and hours, and ascertain the distance the body would describe in half an hour if it began to move from a state of rest.

A velocity of $3\frac{2}{3}$ in. a second is a velocity of $\frac{5}{24}$ of a mile an hour, i.e. at the end of each second the body's velocity is increased by a velocity of $\frac{5}{24}$ of a mile an hour, and therefore in an hour by a velocity of 750 miles an hour. In other words, the acceleration is $3\frac{2}{3}$ when the units are inches and seconds, and 750 when the units are miles and hours. If we obtain the distance described in half an hour from the former value, it will be (by Eq. 2) $\frac{1}{2} \times 3\frac{2}{3} \times (1800)^2$, or 5,940,000 in.; if from the latter value, $\frac{1}{2} \times 750 \times (\frac{1}{2})^2$, or $93\frac{3}{4}$ miles, the same result being obtained in each way.

91. *Accelerative effect of a force.*—When a body moving along a straight line is acted on by a constant force in the direction of the motion, its velocity undergoes a uniform acceleration, and this acceleration is called the accelerative effect of the force. If the force acts in the direction opposite to that of the motion, the velocity undergoes a uniform retardation equal in *amount* to the accelerative effect of the force. Thus, if a body moves downward *in vacuo* near the earth's surface under the influence of gravity, its velocity is uniformly accelerated; the acceleration reckoned in feet and seconds is a little more than 32; the exact number is commonly represented by the letter g, which is called the accelerative effect of *gravity.* In other words, a body falling freely in vacuo

near the earth's surface acquires in each second an additional velocity of g, or a little more than 32 ft. a second. If the body is made to move upward in vacuo under the action of gravity only, so long as the upward motion lasts it loses in each second a velocity of g feet a second. This will continue until it reaches its highest point, when its velocity is zero, and at once it begins to descend.

The nine equations of Art. 89, therefore, apply to the case of bodies moving up or down in vacuo, if we write g (or approximately 32) for f; the first three having regard to a body which is merely allowed to fall, the next three to a body thrown downward with an initial velocity v, the last three to a body thrown upward with an initial velocity v. In all following examples g will be taken as equal to 32 unless the contrary is specified, and this, the student must remember, presupposes that the units employed are *feet* and *seconds*.

It must be added that some forces, such as friction, act always as resistances, and their effect is to retard and never to accelerate motion. When such a force alone acts on a moving body, it will at length bring the body to rest, but will have no tendency to make the body move in the opposite direction to that of its previous motion.

Ex. 106.—If a body were thrown downward with a velocity of 5 ft. a second, and moved for 3 sec. in vacuo, it would describe 159 ft., and would at the end of the 3 sec. be moving at the rate of 101 ft. a second.

Ex. 107.—If a body were allowed to fall in vacuo, the spaces described in the 1st, 2nd, 3rd, 4th, &c., seconds of its motion would be 16, 48, 80, 112, &c., ft.

Ex. 108.—If a body were thrown upward with a velocity of 256 ft. a second, and moved in vacuo, it would at the end of 3 sec. have reached a height of 624 ft., and would be moving at the rate of 160 ft. a second. If the question were asked, what would be the greatest height the body would attain? we may reason thus:—If v is the velocity which the body has when it has reached a height of s ft. above the starting-point, we know from Eq. (9) Art. 89, that $v^2 = (256)^2 - 2 \times 32\,s$; now if s is made to stand

for the greatest height attained, we must have $v=0$, and consequently $s=(256)^2\div 64$, or the body reaches a height of 1024 ft., and then begins to descend. In the case of a force like gravity, it is frequently convenient to use the signs + and − to indicate difference of direction when applying the equations of Art. 89 to particular cases, and to consider s as the distance from the starting-point. How this is done will be made sufficiently plain by the following example.

Ex. 109.—If a body is thrown upward in vacuo with a velocity of 96 ft. a second, at the end of 7 sec. its velocity will be (Eq. (7) Art. 89) $96-32\times 7=-128$, i.e. it is moving *downward* at the rate of 128 ft. a second. From Eq. (8) Art. 89, we see that its distance from the starting-point is $96\times 7-16\times 7^2=-112$, i.e. it is 112 ft. below the starting-point; in fact, it rose to its greatest height of 144 ft. above the starting-point in 3 sec., and fell from it for 4 sec., describing a distance of 256 ft. So that the whole distance described in the 7 sec. is 144 ft. upward and 256 ft. downward.

92. *Relation between momentum and the force producing it.*—By the momentum of a body is meant a quantity proportional to its *mass* and *velocity* jointly. So that if M is the number of units of matter in the body, and V is its velocity, M V is the measure of its momentum. The units used throughout the following pages are pounds, feet, and seconds, e.g. if a body weighs 7 lbs., and moves with a velocity of 12 ft. a second, its momentum is 84.

Suppose a body to weigh 5 lbs. and to move in such a manner that in each second its velocity is increased by 7 ft. per second, its momentum is increased by 35 in each second; in other words, the force, whatever it may be, which produces the uniform acceleration 7, communicates to the body in each second a momentum 35. And this is generally true, if the mass of the body is m, and its motion undergoes a uniform acceleration f, the increase of momentum in each unit of time is mf.

It has been found that the following statements are universally true:—

1. Every change in the velocity or direction of the

motion of a point is due to the action of an external, i.e. an impressed force.

2. The change takes place in the direction of the impressed force.
3. The force is proportional to the change in the momentum of the point.

For the present we shall consider these statements with reference to rectilinear motion. In this case the force acts in the line of the motion, and the change in the momentum is its actual increase or decrease due to the action of the impressed force. When the third of the above statements is applied to comparing two forces, the changes of momentum must be those which take place in the same time. Thus, let F and F_1 denote two forces, the former such as when acting on a mass of m lbs. increases its velocity by f ft. a second in each second, the latter such as when acting on a mass of m_1 lbs. increases its velocity f_1 ft. a second in each second; the forces, therefore, generate in each second the momenta mf and m_1f_1, and consequently we have

$$F : F_1 :: mf :: m_1 f_1.$$

Suppose, for instance, that F acting on a mass of 10 lbs. produces a uniform acceleration of velocity in each second of 12 ft. a second, and similarly that F_1, acting on a mass of 5 lbs., produces a uniform acceleration of velocity in each second of 4 ft. a second, the momenta generated in each second are 120 and 20 respectively, and therefore F is six times as great as F_1.

Two particular cases are included in the above proportion, viz.:—

(a) Suppose the masses m and m_1 to be equal; then

$$F : F_1 :: f : f_1,$$

or, *forces which act on equal masses are in the same ratio as their accelerative effects.*

(*b*) Suppose f and f_1 to be equal; then

$$F : F_1 :: m : m_1,$$

or forces which produce equal accelerative effects on different masses are in the same ratio as the masses.

93. *The absolute unit of force.*—The principles laid down in the last article supply a simple means of measuring force. There will always be a force of a certain magnitude which, acting on a pound of matter, will generate in each second a velocity of one foot per second; if this force is taken as a unit, it is called the absolute unit. Suppose a force F acting on a mass of m lbs. to generate in each second a velocity of f ft. per second, then, by the last article,

$$F : 1 :: mf : 1 \times 1,$$

or

$$F = mf,$$

i.e. if a force, acting on a mass of m lbs., produces an accelerative effect f, estimated in feet and seconds, that force is one of mf absolute units.

Ex. 110.—A mass of 5 lbs. moves from rest under the action of a constant force, at the end of 3 sec. it is moving at the rate of 18 ft. a second, what was the magnitude of the force?

Here the accelerative effect is 6, i.e. the force in each second communicates to the body a velocity of 6 ft. a second; consequently the force must have been one of 5×6, or 30 absolute units.

Ex. 111.—If the force in the last case were to act on a mass of 500 lbs. for a quarter of a minute, what velocity would it communicate to the mass?

If we suppose f to denote the accelerative effect of the force, we have $500f = 30$, or $f = \frac{3}{50}$; consequently, at the end of 15 sec. the velocity will be $15f$, or 0·9, i.e. the body is moving at the rate of 0·9 ft. a second.

94. *Relation between the absolute unit and the gravitation unit.*—It appears from Art. 8 that the gravitation unit is the force exerted by gravity on a pound of matter in London. Now it is found that the accelerative effect of gravity in London is 32·1912 in feet and seconds. Consequently the force of gravity on a pound of matter must be $1 \times 32·1912$, or 32·1912 absolute units. A force, therefore, of P lbs. (i.e. P gravitation units) is *the same as* a force of $P \times 32·1912$ absolute units. We may form a con-

ception of the magnitude of an absolute unit thus:—We see that the gravitation unit contains rather more than 32 absolute units; if, therefore, we were to fasten a weight of half an ounce to the end of a thread, the force required to hold the weight suspended is very slightly more than an absolute unit.

The student's attention is particularly directed to the following point:—The numerical values of the accelerative effect of gravity at various points of the earth's surface differ slightly, but sensibly, from each other; consequently the force of gravity on a pound of matter will have one value at one place, another at another; these different values are all *about* 32 absolute units. In very many questions, to take the value as 32, without reference to place, gives a sufficiently close approximation to the required result. If, however, it were possible to get to a place sufficiently above the earth's surface, this approximation would no longer hold good. For instance, at a place where the accelerative effect of gravity is reduced to 25, the force of gravity on 10 lbs. of matter is 250 absolute units, whereas in London it is as much as 321·912 absolute units. If M is the mass of a body, and g the accelerative effect of gravity at any place, the force of gravity on the mass at that place is Mg absolute units.

95. *Motion on a smooth horizontal plane.*—Suppose a body to move along a straight line, and that we know its velocity at any given instant; now if its motion is uniformly accelerated or retarded, and if we know the amount (f) of the acceleration or retardation per unit of time, it is plain that we know all the circumstances of the motion, and can calculate, for instance, the space described and the velocity acquired in a given time by Art. 89. In this and the next few articles we will show how to determine the accelerative effect in certain simple cases of motion, and first we will take the case of motion on a smooth horizontal plane:—A B represents such a plane; a body whose mass is M is moved along it by a horizontal force P. The only forces acting besides P are the weight of the body and the reaction of the plane, and these are in equilibrium. Hence f is given by the equation

FIG. 91.

$$P = Mf.$$

If M is given in pounds and P in absolute units, f will be in feet and seconds. If P is given in gravitation units, it must be multiplied by 32·1912, or approximately by 32, in order to reduce it to absolute units.

96. *Motion on a rough horizontal plane.*—The forces are the same as in the last case, except that there is in addition the friction between the plane and the body, and this force will act so as to oppose the motion. Let μ denote the coefficient of friction, M the mass of the body in pounds, its weight in absolute units will be Mg, and consequently the friction will be $Mg\mu$ in absolute units. Hence if P is measured in absolute units, the

force producing motion is $P - Mg\mu$; and therefore f is given by the equation

$$P - Mg\mu = Mf.$$

If P is less than $Mg\mu$, the motion will be uniformly retarded.

Ex. 112.—Let M be a mass of 100 lbs., the coefficient of friction between it and the plane 0·1, P a force of 12 gravitation units, and let it be required to determine the velocity of the body after it has moved from rest for 3 sec. Supposing the accelerative effect of gravity at the place to be 32.

P is a force of $12 \times 32{\cdot}1912$, or $386{\cdot}2944$, (say) $386{\cdot}3$ absolute units. Friction is a force $100 \times 32 \times 0{\cdot}1$, or 320 absolute units; therefore we have a mass of 100 lbs. moved by a force of $386{\cdot}3 - 320$, or 66·3 absolute units. If, then, f is the accelerative effect of the force,

$$66{\cdot}3 = 100f,$$

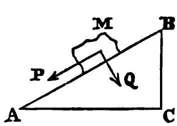
or

$$f = 0{\cdot}663,$$

and the velocity of the body at the end of 3 sec. will be $3f$, or 1·989, or very nearly 2 ft. per second.

The student should attend to the following points:—(1) If instead of 32·1912 we use the approximate number 32, we should have $f = 0{\cdot}64$, and the required velocity 1·92 ft. a second. (2) If it were possible for the plane and the body to be at a place where the accelerative effect of gravity is reduced to 24, the friction would be reduced to 240 absolute units, so that the force producing motion would be $386{\cdot}3 - 240$, or 146·3 absolute units, and the required velocity would be 4·389, or nearly 4·4 ft. a second.

FIG. 92.



97. *Motion on a smooth inclined plane.*—AB is the smooth inclined plane; its length, height, and base may be denoted l, h, and b. M is the mass of the body sliding down the plane. The forces acting on the body are the reaction of the plane and gravity, the latter being a force of Mg units acting vertically downward, and is equivalent to two forces, P and Q, acting along and at right angles to the plane respectively, where (see Ex. 58)

$$P = Mg\,\frac{h}{l} \qquad \text{and} \qquad Q = Mg\,\frac{b}{l}.$$

Now Q is balanced by the reaction of the plane, and thus P is left unbalanced and produces the motion; if then f be the acceleration, we have

$$P, \text{ or } Mg\,\frac{h}{l} = Mf,$$

and therefore

$$f = \frac{gh}{l}.$$

Ex. 113.—If the length of the plane is 10 times the height, the acceleration of the motion of a body sliding down the plane (supposed to be smooth) is $g \div 10$, or 3·2 (approximately) in feet and seconds. If it is asked how long it will take a body to slide down 40 ft. of the length of the plane starting from rest, the formula $s = \frac{1}{2} f t^2$ must be used, and it will be found that the time is 5 sec.

98. *Motion on a rough inclined plane.*—In the last article, if the plane is rough and μ denotes the coefficient of friction between the body and the plane, there will be an additional force of friction equal to $\text{M} g \mu b \div l$, acting along the plane in a direction opposite to the motion, and consequently f will be determined by the equation

$$\text{M} g \frac{h}{l} - \text{M} g \mu \frac{b}{l} = \text{M} f,$$

or

$$f = \frac{g(h - \mu b)}{l}.$$

If μb is greater than h, the motion will be uniformly retarded; if the body moves up the plane, the motion will be retarded both by gravity and friction.

Ex. 114.—Let the coefficient of friction between the body and the plane be 0·2, the length of the plane three times the height, consequently the base of the plane will be $\sqrt{\frac{8}{9}}$, or 0·943 times the length. Hence the resolved part of gravity acting along the plane will be $\frac{1}{3} \text{M} g$ in absolute units, and the perpendicular pressure on the plane 0·943 $\text{M} g$ absolute units, and therefore the friction will be $0{\cdot}2 \times 0{\cdot}943 \, \text{M} g$, or 0·1886 $\text{M} g$; consequently the force urging the body down the plane is $(0{\cdot}3333 - 0{\cdot}1886)$ $\text{M} g$, or 0·1447 $\text{M} g$. If, then, f is the accelerative effect, this force must equal $\text{M} f$, and therefore

$$f = 0{\cdot}1447 \, g.$$

If we take g approximately at 32, we have $f = 4{\cdot}63$ in feet and seconds. If we were required to find the velocity acquired by the body in sliding down 100 ft. of the length of the plane, we shall have, by Art. 89,

$$v^2 = 2 \times 4{\cdot}63 \times 100,$$

or at the end of the 100 ft. the body is moving at the rate of 30·4 ft. a second.

Ex. 115.—In the last case suppose the material of the body to be such that the coefficient of friction between it and the plane is 0·5. Then the force of friction is $0{\cdot}5 \times 0{\cdot}943 \, \text{M} g$, or 0·472 $\text{M} g$. As this is greater than 0·333 $\text{M} g$, the body will remain at rest; but if it is made to slide down the plane, its motion will be uniformly retarded by a force of $(0{\cdot}472 - 0{\cdot}333) \, \text{M} g$, or 0·14 $\text{M} g$. Suppose the body is at any instant sliding down the plane at the

rate of 20 ft. a second, and it is asked how long it will continue to move. If f is the retarding effect of the force, we have $f=0{\cdot}14\,g$, or $4{\cdot}5$ in feet and seconds (very nearly). Now if v is the velocity of the body after it has moved for t sec., we have

$$v=20-4{\cdot}5\,t.$$

Consequently the time at which the body comes to rest will be given by the equation

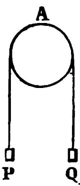
$$0=20-4{\cdot}5\,t,$$

or $$t=4\tfrac{4}{9};$$

i.e. the body will move for $4\frac{4}{9}$ sec., and will then come to rest. It will, of course, follow from the nature of friction that the body will stay at rest, and not turn back upon its course, as a body does when thrown upward.

99. *Motion of two bodies connected by a thread passing over a smooth point.*—A is a fixed smooth block or point, over which a perfectly flexible thread passes, the weight of which can be neglected; P and Q are weights fastened to the ends of the thread. Supposing P to be greater than Q, we have the whole mass, P + Q, put in motion by the excess of the force of gravity on P over that on Q; hence the force producing motion is $Pg-Qg$, and consequently

FIG. 93.

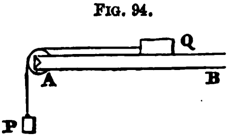


$$(P+Q)f=(P-Q)\,g.$$

Ex. 116.—If P and Q are respectively $8{\cdot}5$ and $7{\cdot}5$ lbs., then $16\,f=1\times g$ or $f=2$ (approximately). Let it be asked what distances will P (supposed to start from rest) describe in 1, 2, 3, ... sec. We apply the formula $s=\frac{1}{2}ft^2$, and find that it describes 1 ft. in the first second, 3 in the second, 5 in the third, and so on.

100. *Motion of two bodies connected by a thread, one falling vertically and drawing the other along a horizontal table.*—A B is the table on which the mass Q is placed; P, which falls vertically, is connected with Q by a thread, which passes over a perfectly smooth point at A—the thread is supposed to be perfectly flexible and without weight. The table is supposed to be rough, and μ is the coefficient of friction between it and Q. Now the whole mass moved is P + Q, and the force producing motion is the excess of the weight of P (viz. Pg) over the friction (viz. $Qg\mu$).

FIG. 94.



Hence $$(P+Q)f=(P-Q\mu)\,g.$$

Ex. 117.—If P and Q are masses of 20 and 100 lbs. respectively, and if the coefficient of friction between Q and the table is $0{\cdot}16$, we have a mass of 120 lbs. moved by a force of $(20-100\times 0{\cdot}16)\,g$, or 128 absolute units, and therefore $f=1\frac{1}{15}$. If P fell through $4{\cdot}8$ ft., we find by the formulas $v^2=2fs$ and $v=ft$ that the system would acquire a velocity of $3\frac{1}{5}$ ft. a second, and be 3 sec. in describing the distance.

101. *Experimental verification of the laws of rectilinear motion.*—It might be thought at first sight that it would be easy to verify these laws by direct experiments on falling bodies; but when this point is considered with a view to performing the experiments, there are found to be two chief difficulties which are practically very serious; the first is that which arises from the resistance of the air, the second is that of measuring short intervals of time. We will premise a word or two on these points:—

(*a*) The resistance offered by the air to the motion of a dense body moving with a moderate velocity is small, but it becomes very considerable when the velocity is as large as that which a body acquires in falling for several seconds. This is well illustrated by Newton's experiments made in the cathedral church of St. Paul, where bodies could be let fall through a clear height of 220 ft. He used two sets of glass balls, one about $\frac{3}{4}$ in. in diameter filled with mercury, and weighing about 2 oz., the other about 5 in. in diameter filled with air, and weighing about $1\frac{1}{2}$ oz. There were six balls in each set, and he found that on the average the first six fell in about 4 sec., while the second six fell in rather more than 8 sec.[1] This difference was wholly due to the greater resistance offered by the air to the motion of the larger and lighter body.

(*b*) By equal times we mean the times occupied by exactly similar motions. Suppose, then, that a pendulum supported on a fine steel edge is set swinging; it will be observed that its arc of vibration diminishes very slowly, so that several successive vibrations are as nearly as possible similar motions, and are therefore performed in equal times. To overcome the difficulty of counting these vibrations, clockwork consisting of wheels, an escapement, and a falling weight may be added, so that the completion of each vibration may be marked by a distinct tick, and then the time can be noted by the ear, while the eye watches the motion of the body. But whether the machinery be added or not, the difficulty of counting the vibrations while the motion of the body to be observed is in progress can be overcome by practice; e. g. with practice an observer can tell that a certain body moved over a certain distance while the pendulum made $3\frac{1}{2}$ vibrations. The first difficulty, then, will be overcome if we experiment on bodies moving with moderate velocities. The second may be overcome by practice in using a pendulum.

Experiment 1.—Take two or three balls of various sizes and of different metals or other dense substances, and let them fall at the same instant from any high place, e. g. the top floor of a house; they will be found to strike the ground together, and this for all heights, whether 20, 30, or 40 ft.; consequently all their velocities must undergo the same acceleration at all parts of their motion. The force of gravity upon them must therefore be proportional to their masses (Art. 92, *b*). That this verification of this fundamental principle could be extended to bodies of small density but for the resist-

[1] *Principia*, lib. ii. prop. xl. Schol. Exp. 13.

tance of the air, is illustrated by the well-known experiment of allowing a feather and a piece of metal to fall along the length of a tube from which the air has been withdrawn. If care is taken, they can be let fall at the same instant, and then they reach the bottom of the tube together.

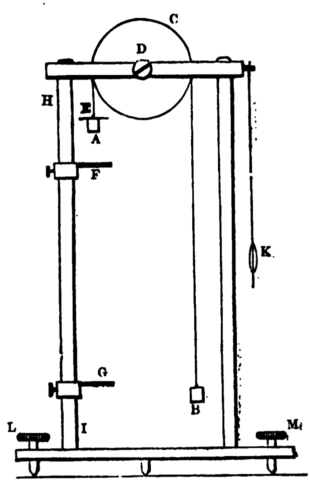
Experiment 2.—Take a long board so thick as not to bend sensibly in the middle, and cut a shallow groove along its length. Set it at a small inclination to the horizon. A small heavy ball will roll along the groove with but little resistance from friction and air. Set a pendulous body swinging, and mark two points O and A, such that the body rolls from O to A in one swing of the pendulum. Mark a second point B such that the body rolls from O to B in two swings of the pendulum. If sufficient care is taken, it will be found $OB = 4 . OA$. We may change the inclination of the board, this will change the positions of A and B, but we shall still have $OB = 4 . OA$. Again we may change the length of the pendulum, this will change the time of one swing, and therefore will change the positions of A and B, but still OB will equal $4 . OA$. This result is consistent with the rule that the space described is proportional to the square of the time of description ($s = \frac{1}{2} f t^2$, or $s \propto t^2$). And the rule may be further confirmed by comparing the spaces described in two and three swings of a pendulum. The conclusion from this series of experiments is that the force which causes the body to move down the plane has a constant accelerative effect, i.e. in the case of the inclined plane f is constant. Now as $g = f l \div h$ (Art. 97), g must also be constant, i.e. sensibly constant at a given place on the earth's surface, and therefore that if the motion of bodies falling in a vacuum could be observed, they would follow the rules stated in Art. 89.[1]

Experiment 3.—*Atwood's Machine*, which is shown in the annexed figure :—A, B are two brass boxes which can be made heavier or lighter by putting more or fewer small shot into them, but which under all circumstances are of the same weight; they are connected by a fine silk thread A C B, passing over a pulley C, which may be supported by the points of two screws, the head of one of which is seen at D. More commonly, however, the axis of the pulley rests on two pairs of friction wheels, so that the rotation of the axis may take place with a rolling instead of a sliding motion (Art. 71, *b*). Friction being thus reduced to a very small force, it follows that for moderate velocities the boxes and wheel are very nearly an inert mass, which may be set in motion when a preponderance is given to A by means of a flat weight or bar placed on it. F is a ring through which the box A can pass, but which stops the flat weight or bar; consequently after

[1] Experiments 1 and 2 were originally made by Galileo. In the case of the second experiment he compared the times in which the body described the distances OA and OB by taking a vessel full of water, making a small hole in the bottom, and measuring the quantity of water that flowed through while the body was describing the several distances. The times are, of course, proportional to the quantity of water.

A passes through F the machine moves under the action of balanced forces, and its motion will be uniform. G is a small stage which arrests A in its descent, and stops the motion. A second or half-second pendulum K is part of the apparatus, and by means of it the time of motion of the boxes can be noted with considerable accuracy. There is a graduated scale on H I, and the instrument can be levelled by means of three levelling screws. Suppose that A begins to move from H, and that F is adjusted by trial to such a position that A passes through it at the end of 3 half-sec., and then that G is adjusted in such a position that A reaches it in 5 more half-sec. Suppose that F G is 6 ft.; as the distance F G is described with a uniform velocity, it follows that that velocity is $6 \div 2\frac{1}{2} = 2 \cdot 4$ ft. a second. Now this is the velocity which A has acquired on reaching F, i.e. the system has acquired in $1\frac{1}{2}$ sec. a velocity of 2·4 ft. per second. Now suppose the experiment repeated, all the circumstances being the same, except that F is so placed that A reaches it at the end of 4 half-sec.; it would be found tha A reaches it with a velocity of 3·2 ft. a second. Now it will be observed tha

FIG. 95.



$$2 \cdot 4 : 3 \cdot 2 :: 3 : 4;$$

in other words, the velocities acquired are proportional to the times, and therefore the motion has been uniformly accelerated. In this case the numerical value of the accelerative effect (f) is $2 \cdot 4 \div 1\frac{1}{2} = 1 \cdot 6$. The force by which this acceleration has been produced is the action of gravity on the bar; we will denote the weight of the bar by w.

We can use this machine to verify principle b of Art. 92. Thus, suppose a bar whose weight is w_1 to be used, and the weights of the boxes to be adjusted so that there is no change in the whole mass moved, and suppose that the accelerative effect (f_1) is determined as before, it will be found that

$$w : w_1 :: f : f_1.$$

Now we have already seen (Exp. 1) that the forces producing the motion are proportional to the masses w and w_1; the above proportion is therefore a particular case of that given in principle b, Art. 92, of which it is a direct verification.

We can also use the machine to verify the general principle of Art. 92.

Thus, the whole mass moved is that of the bar (w), that of the two boxes (W, W), and that of the pulley, the mass of the thread being neglected; now as the pulley has a rotatory motion, the same velocity is not communicated to all its parts; consequently its effect is not exactly the same as its mass, but as that of a certain mass M, which stands in a known relation to its mass. The whole mass moved may therefore be taken to equal $2W+M+w$, and, if f is the acceleration of the motion of the boxes determined as above $(2W+M+w)f$ is the momentum produced in a second by the preponderating force. If now we change W into W_1, and w into w_1, and in consequence f is changed into f_1, it will be found that the forces producing motion are in the ratio

$$(2W+M+w)f : (2W_1+M+w_1)f_1,$$

i.e. the forces are proportional to the momenta they generate in a unit of time.

We may also use the machine for determining the numerical value of g. For since the force producing motion is wg absolute units, we must have

$$wg=(2W+w+M)f,$$

and with proper care an approximate value of g can be found from this equation.

Experiment 4.—*Morin's Machine.*—Let $ABCD$ be a flat board placed in a vertical plane, and P a body to which is fastened a pencil or brush dipped in Indian ink. Suppose that P is allowed to fall in front of the board in such a manner that the pencil point touches it, it is plain that a vertical line will be traced on the board. Now suppose that by any means the board is made to move forward with a uniform velocity, so that by the time P has fallen to Q, the board has come into the position $A'B'C'D'$; then the point originally covered by P will have advanced to O, and, instead of a vertical straight line, a curved line OQ will have been described. If V is the velocity of the board, $OP \div V$ will be the time in which the body falls through the vertical height PQ. The same will be true of all points of the curve; e.g. let OH be the curve actually drawn; draw the horizontal and vertical lines OK, KH; the board must have moved through the horizontal distance OK, while the body was falling through the vertical height KH. Similarly, if any point of the curve (L) be taken, and a perpendicular (LM) be drawn to OK, the board would move through a horizontal distance MO while the body falls through a vertical height ML. Now, supposing the curve to be correctly drawn, we can use *it to show that* the accelerative effect of gravity is constant; thus, using

FIG. 96.

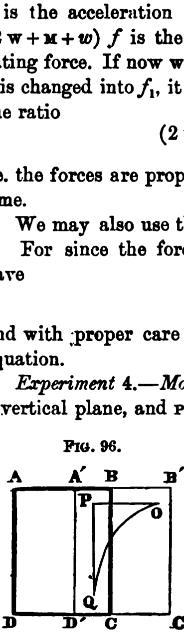
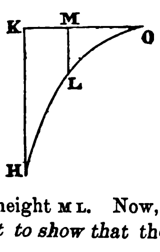


FIG. 97.



the formula $s=\frac{1}{2} f t^2$, and observing that $t=\text{K O}\div\text{V}$ when $s=\text{K H}$, we have

$$\text{K H}=\frac{f}{2\,\text{V}^2}\cdot\text{K O}^2, \qquad (1)$$

and similarly

$$\text{M L}=\frac{f}{2\,\text{V}^2}\cdot\text{M O}^2. \qquad (2)$$

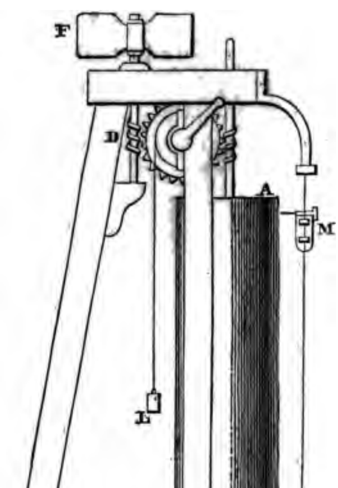
If then f is the same throughout the motion, we must have for all points of the curve

$$\text{K H} : \text{M L} :: \text{K O}^2 : \text{M O}^2.$$

And this is found to be the case, e. g. if $\text{M O}=\frac{1}{3}\cdot\text{K O}$, M L is found to be $\frac{1}{9}$th of K H. The actual value of f (viz. 32) can evidently be determined from either (1) or (2), if V is known correctly.

What has been said above describes the principle of Morin's machine; in the machine, however, it is found convenient to substitute a cylinder covered with a sheet of paper for the board. The actual arrangement is shown in the annexed figure. M is the weight carrying the pencil, guided in its fall by two threads M N, one of which hides the other; the point is thus kept in contact with the cylinder A B. L is a heavy weight fastened to the end of a rope wound round a drum, to which is fastened a toothed wheel working with an endless screw cut on the axis of the cylinder. L in its descent thus turns the cylinder. The motion is rendered uniform by causing the wheel to work with an endless screw D on the axis of a fly F. When L begins to fall, its velocity and the velocities of the fly and cylinder are accelerated; but the resistance of the air, increasing rapidly with the velocity, soon stops the acceleration of the velocity of the fly, and therefore of L and of the cylinder, and then the velocity becomes uniform. When this state has been produced, M is allowed to fall, and the curve is drawn. The velocity (V) of a point on the cylinder is thus found:—Suppose its circumference to be 3 ft., and that it makes 40 revolu-

FIG. 98.



tions in half a minute; plainly 40×3 ft. are described in 30 sec., and therefore $v = 40 \times 3 \div 30 = 4$ ft. a second.

A similar method can be applied to observing other motions; thus, in order to ascertain whether it is true that the coefficient of friction between two substances is independent of the velocity with which one moves over the other (Art. 12, 3), experiments were made as follows. A mass Q is placed on a horizontal table (fig. 94), and caused to slide along the table by means of a falling weight P, as in Art. 100. To P is attached a brush dipped in Indian ink, and placed in contact with the cylinder of one of the above machines. The curve obtained is found in all cases to be such as to prove that the motion undergoes a uniform acceleration, and therefore is produced by a constant force. Now the force producing motion is $P g - Q g \mu$; and as P, Q and g are known to be constant, μ—the coefficient of friction between Q and the table—must also be constant.

102. *Energy or accumulated work.*—When a heavy point is in motion, it has in it a capacity of overcoming a resistance through a certain space. Suppose the point is moving at any instant with a velocity V, and that from that instant its velocity undergoes a uniform retardation (f), it will come to rest after describing a distance s, given by the equation (Art. 89, Ex. 103)

$$V^2 = 2 f s.$$

Let M be the mass of the heavy point, P the resistance which produced the retardation, we have

$$P = M f,$$

and therefore $$\tfrac{1}{2} M V^2 = P s.$$

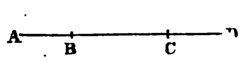
Now P s is the work done against the resistance while the point is being brought to rest; consequently the point, in virtue of its mass and velocity, was capable of doing an amount of work measured by P s or $\frac{1}{2} M V^2$; this is therefore called the *energy*, or *accumulated work*, or *vis viva* of the point when moving with that velocity. If we suppose V to be estimated in feet and seconds and M in pounds, P will be in absolute units and s in feet. Let us reckon as a unit of work the work done by an absolute unit of force when its point of application moves through a foot

in the direction of the force (Art. 68); then $\text{P}\,s$, or $\frac{1}{2}\,\text{M}\,\text{V}^2$, is the number of units of work which measures the energy. It is plain from Art. 94 that to obtain the force in gravitation units we must divide P by 32·1912, or approximately by 32, and consequently the energy in foot-pounds is $\text{M}\,\text{V}^2 \div 2\,g$, where g denotes 32·1912.

Ex. 118.—Let M be a mass of 20 lbs., and V a velocity of 28 ft. a second; then $\frac{1}{2}\,\text{M}\,\text{V}^2 = 7840$, i.e. the body, while moving with that velocity, has an energy measured by 7840 units of work; and if at any instant its velocity were observed to diminish and at length to cease, it would have done, while being brought to rest, 7840 units of work against the resistance or resistances by which its velocity was destroyed. If the energy were estimated in foot-pounds, its value would be 243·5, or approximately 245.

103. *Why energy is called accumulated work.*—If we suppose the body to be moved from rest, and to have a velocity V gradually communicated to it by the action of any force, it can be easily shown by reasoning similar to that used above that $\frac{1}{2}\,\text{M}\,\text{V}^2$ units of work must have been done on the body by the force while communicating the velocity. From this point of view, it is easy to see why the energy should be called accumulated work. For the body is at a given instant moving with a velocity V, and we can make two assertions about it—*first*, while the velocity was gradually increased from O up to V, the force or forces producing the velocity must have done on the body $\frac{1}{2}\,\text{M}\,\text{V}^2$ units of work, and these remain accumulated in the body; for, *secondly*, if the velocity is gradually diminished from V down to O by the action of a resistance or resistances, the body, in virtue of its energy, will do against the resistances $\frac{1}{2}\,\text{M}\,\text{V}^2$ units of work.

FIG. 99.



Ex. 119.—Let A, B, C, D be four points in a straight line, and suppose a body with a mass of 12 lbs. to begin to move from rest at A, and on reaching B to have acquired a velocity of 44 ft. a second. Suppose it to retain this velocity unchanged between B and C, and between C and D to lose its velocity gradually

till it comes to rest at D. Now, observing that in this case $\frac{1}{2}$ M V^2 equals 11,616, we can assert (*a*) that between A and B the body was under the action of a force or forces which did 11,616 units of work; (*b*) that between B and C the body was under the action of balanced forces, and that it had an energy or capacity for doing 11,616 units of work; and (*c*) that between C and D it actually did that number of units of work against some resistance or resistances. If we know the distance A B, and that the force acting along A D on the body was constant, we can determine it; or if we know the force, we can determine the distance A B. Thus, if the distance from A to B were 22 ft., the body must have been acted on by a force of 11,616 ÷ 22, or 528 absolute units, while describing the 22 ft., provided the force were constant. Again, if the body were brought to rest by the action of a constant force of 192 absolute units, while moving from C to D, that distance must be 11,616 ÷ 192, or 60·5 ft.

Ex. 120.—A cubic foot of iron (whose mass is 450 lbs.) falls from a height of 6 ft. on to a very powerful spring which yields through a distance of 1-10th of an inch; required the force (supposed to be constant) exerted by the spring.

Since the weight of the body is (approximately) a force of 450 × 32 absolute units, we see that during the fall gravity did 450 × 32 × 6, or 86,400 units of work. This is the energy of the body at the instant it touches the spring, and is wholly employed in forcing the spring back through the 1-120th of a foot. Hence if P is the force (supposed constant) with which he spring resists compression, we must have

$$P \times \frac{1}{120} = 86,400,$$

and, therefore, P equals 10,368,000 absolute units or (approximately) 324,000 gravitation units. It will be observed (*a*) that the resistance offered by the spring is in reality not a constant force, and consequently that the value obtained for P is only a sort of mean value of the force actually exerted; (*b*) that the 6 ft. ought in strictness to include the distance through which the spring yields; (*c*) that a very different result would be obtained if the motion took place where the force of gravity differed much from 32, e. g. if the motion could be observed at a place where the force of gravity is 20, P would be 6,480,000 absolute units, or 202,500 gravitation units approximately, or 201,297 gravitation units exactly.

Ex. 121.—A body weighing W lbs. slides down a rough inclined plane; the coefficient of friction between the plane and the body is 0·2; the slope is 2 vertical to 5 horizontal; the base is 50 ft. long. Find the velocity acquired by the body in sliding down the length of the plane.

The force of gravity on the body is Wg absolute units; so that the work done by gravity while the body descends vertically through 20 ft. is Wg × 20 units, and the work expended on friction is 0·2 Wg × 50 units (*Art. 72*). *Hence* the energy of the body at the foot of the inclined plane

is $W g \times 20 - 0{\cdot}2\, W g \times 50$, or $10\, W g$ units. But if V is the velocity of the body at the foot of the incline, the energy is $\frac{1}{2}\, W V^2$ units; so that $\frac{1}{2}\, W V^2 = 10\, W g$, i.e. V^2 equals $20\, g$, and V equals $\sqrt{640}$, or 25·3 ft. per second, if we suppose g to equal 32.

104. The case in which the velocity undergoes a change, without the body being at rest either at the beginning or end of the motion, follows immediately from what has gone before. If M is the mass of the body, V its velocity at the beginning and v at the end of any interval of time, the energy at the beginning of the interval was $\frac{1}{2}\, M\, V^2$, and at the end $\frac{1}{2}\, M\, v^2$; therefore the increase during the interval was $\frac{1}{2}\, M\, (v^2 - V^2)$. Let us suppose this to have been produced by the action on the body of a constant force P acting in the direction of the motion while the body describes a distance s; we shall have

$$\frac{1}{2}\, M\, (v^2 - V^2) = P\, s.$$

If we suppose M to be estimated in pounds, and v and V in feet per second, we shall have s in feet, and P in absolute units. Hence, if we wish to estimate P in pounds (gravitation units) and the energy in foot-pounds, we must divide by 32·1912, or approximately by 32.

Ex. 122.—A train weighs W lbs.; it comes to the foot of an incline of 1 in 140 with a velocity of 30 miles an hour, and runs up merely by its energy; supposing the friction to be 8 lbs. a ton, how far must it go along the incline that its velocity may be reduced to 15 miles an hour? Let x denote the horizontal distance in question, then the train is raised through a vertical height of $\frac{x}{140}$ ft., and therefore there will be $\frac{x}{140} \times W g$ units of work expended on gravity; also, as the co-efficient of friction is $8 \div 2240$, or $1 \div 280$, there will be $\frac{1}{280}\, W g \times x$ units of work expended on friction. Now bearing in mind that a velocity of 30 miles an hour is the same as 44 ft. a second, we have

$$\frac{1}{2}\, W\, (44^2 - 22^2) = \frac{1}{140} \times x\, W g + \frac{1}{280} \times x\, W g.$$

Or, assuming g to equal 32,

$$x = 2117{\cdot}5 \text{ ft.}$$

105. *Most general relation between the power and the weight in any machine.*—The machine will consist of a

number of portions of matter, each moving at a given instant with a certain velocity. Suppose the mass of each portion to be multiplied by the square of its velocity and the whole to be added together, one-half of this sum is the energy of the machine at that instant. Suppose the energy to be determined in a similar manner at a subsequent time. If we subtract the first determination from the second, we obtain the change of the energy of the machine during the interval. During this interval work will be done against the passive resistances, friction, &c. Also if any part of the machine undergo elongation, compression, or distortion, work will be done against the internal or molecular forces. Now the fundamental principle we have to state is this:—The work done by the power at the driving point of the machine during any interval equals the work done against the weight at the working point of the machine, together with (a) the change in the energy of the machine, (b) the work done against friction, &c., (c) the work done against the molecular forces.

It must be understood that if the 'power' is a mass, or the 'weight' a mass, the change in their energy is to be reckoned as part of the change in the energy of the machine.

We will illustrate this principle by two simple cases:—

Ex. 123.—In Art. 99, if P and Q are masses of 20 and 18 lbs. respectively, what velocity would they acquire while P falls through 12 ft.?

If v denotes the velocity of the bodies, the energy of the whole system is $\frac{1}{2} \times 20\, v^2 + \frac{1}{2} \times 18\, v^2$, or $19\, v^2$. The number of units of work done by P will be $12 \times 20\, g$, and the number expended in raising Q will be $12 \times 18\, g$. Hence

$$12 \times 20\, g = 19\, v^2 + 12 \times 18\, g,$$

or
$$19\, v^2 = 24\, g;$$

therefore
$$v = 6{\cdot}36 \text{ ft. per second.}$$

Ex. 124.—In Art. 100, suppose P and Q to weigh 12 lbs. and 80 lbs. respectively, and the coefficient of friction between Q and the table to be 0·1. Find the velocity acquired when P falls through 8 ft.

If v denotes the common velocity of P and Q, their energies will be

$\frac{1}{2} \times 12 v^2$ and $\frac{1}{2} \times 80 v^2$, and the whole energy will be $46 v^2$. Now the work done by P in falling 8 ft. is $8 \times 12 g$ units, and that expended on Q, $8 \times 0{\cdot}1 \times 80 g$.

Hence $$8 \times 12 g = 46 v^2 + 8 \times 0{\cdot}1 \times 80 g;$$

therefore $$v = 4{\cdot}72 \text{ ft. per second.}$$

106. *The mutual action between two moving bodies.*—We have already seen (Art. 10) that when a body A acts on a second body B, the body B reacts with an equal opposite force on A. Moreover we have already considered several cases in which the system in motion consists of two connected bodies, and we have been able in these cases to investigate the motion of both considered as forming one system. We will now enquire what is the magnitude of the mutual actions between the two bodies. Since the motion of the system is known, the motion of either of the bodies (say of A) is known, and therefore the force producing that motion can be determined. Now this force must be the resultant of known forces and of R, the required reaction of B on A; consequently we have the means of determining the force R. In the actual determination of R we may reason either on the acceleration of A's velocity, or on the change in A's energy; in fact, any effect due to the action of R will serve to determine that force. Commonly the determination of the mutual action and reaction between the parts of a connected system of bodies in motion is a matter of considerable difficulty; in a few cases, however, the difficulty is but small, and some of these we will now discuss.

Ex. 125.—In Art. 99, let P and Q be masses of 20 lbs. and 15 lbs.; find the mutual action between the bodies.

Here, if f is the acceleration of P's motion, we know that $(20+15) f = (20-15) g$, or $f = \frac{1}{7} \times g$. Now the forces acting on P are gravity downward, and Q's action upward. If, therefore, R denotes the latter force, the acceleration of P's motion is produced by $20 g - R$, and consequently

$$20 g - R = \tfrac{1}{7} \times g \times 20.$$

Therefore $$R = \frac{120 g}{7}, \text{ or } 548\tfrac{4}{7} \text{ absolute units.}$$

In the same manner P's action on Q can be shown to equal $120\,g \div 7$ absolute units. It will, of course, follow that the thread is subjected to a tension of $548\frac{4}{7}$ absolute units or (approximately) $17\frac{1}{7}$ lbs. (gravitation units).

We might also reason in this case as follows:—Since the motions of P and Q are known to be uniformly accelerated, and since the forces acting on one of them (say P) are gravity and the action of Q, that action must be a constant force; let it be denoted by R. Suppose P to fall through a certain distance from rest, say h ft., then the work done on P must be $(20\,g - R)\,h$, and this must equal the energy of P, i.e. $(20\,g - R)\,h = 10\,v^2$, if v denotes P's velocity. In the same manner by reasoning on Q's motion we obtain the equation

$$(R - 15\,g)\,h = 7{\cdot}5\,v^2;$$

and then, if we multiply these equations crosswise,

we have $$10\,(R - 15\,g) = 7{\cdot}5\,(20\,g - R),$$

or $$7\,R = 120\,g,$$

the same result as that obtained before. The student must bear in mind that, unless it is known that R is a constant force, the above reasoning would have to be modified; the product $R\,h$ is the work done by R on the supposition that R is constant. If R is increasing or decreasing in some known way, there are methods by which the work done by R, when acting through a known distance, can be found.

Ex. 126.—In Art. 100 determine the mutual action between P and Q; if P and Q are masses of 20 and 100 lbs. respectively, and there is no friction between Q and the table.

If f is the acceleration of Q's motion, we have

$$(20 + 100)f = 20\,g;$$

and if R is P's action on Q transmitted along the thread, we have

$$R = 100f.$$

Hence $$R = \frac{100\,g}{6}, \text{ or } 533\tfrac{1}{3} \text{ absolute units.}$$

Or we may reason thus:—Since the motion of the system is uniformly accelerated, R must be constant, and then, as in the last example, we can form the equations—

From P's motion, $$(20\,g - R)\,h = 10\,v^2;$$

from Q's motion, $$R\,h = 50\,v^2;$$

therefore $$R = \frac{100\,g}{6}.$$

107. *The force of inertia.*—By the *inertia* of a body is *meant* its tendency to continue in its present state of rest

or of uniform motion in a straight line, unless that state is changed by the action of an external force or forces. Now consider a material point (A), and suppose its state of rest or of uniform motion in a straight line to be changed in any way by the action (R) of a second body (B), then, as we have already seen (Art. 10), A will exert on B a reaction equal and opposite to R. This reaction which A exerts on B is the force of A's inertia. Thus, in Ex. 126 the body Q is an inert mass of 100 lbs.—the force of gravity on it being exactly neutralised by the reaction of the table; its motion undergoes a uniform acceleration from the action of the falling body P. This action, which is transmitted along the thread, is, as we have seen, a force of $533\frac{1}{3}$ absolute units, and the reaction of Q on P likewise transmitted along the thread is a force of $533\frac{1}{3}$ absolute units; this force which Q exerts on P is the force of Q's inertia.

108. *Case of two masses acted on by equal forces.*—We shall see in what follows that several interesting questions arise out of cases in which we know that two bodies have been acted on during equal times by equal forces, but in which more is not known about the forces. In these cases we can draw two inferences:—

(*a*) The momenta imparted to the bodies are equal.

(*b*) The energies imparted are inversely as the masses.

Suppose that P and Q are the masses of the bodies which are acted on by equal forces R, for a time t. If f and f' are the accelerations due to the action of R, and u and v the velocities communicated during the time t, we know (Art. 89) that

$$u = ft \quad \text{and} \quad v = f't,$$

or

$$\text{P}u = \text{P}ft \qquad \text{Q}v = \text{Q}f't.$$

But we know (Art. 92, 3) that

$$\text{P}f = \text{R} \quad \text{and} \quad \text{Q}f' = \text{R};$$

consequently

$$\text{P}u = \text{Q}v;$$

i.e. the momenta imparted to the bodies are equal. By squaring the above equation we obtain

$$P^2 u^2 = Q^2 v^2,$$

and therefore $\frac{1}{2} P u^2 : \frac{1}{2} Q v^2 :: Q : P,$

i.e. the energies are inversely proportional to the masses. It must be observed that these conclusions are equally true whether the equal forces R continue constant throughout the motion or not; all that is necessary is that the forces should be equal to each other at each instant. We have only to suppose the time divided into a number of small parts, during each of which the forces are constant, and the conclusions above stated will then hold good for each of these short intervals, and so for them all when put together.

109. *Motion of a shot within the bore of a cannon.*—The case considered in the last article would be exemplified if a spring were placed between two bodies, and then allowed to expand. In principle this example is the same as that of a gun, and the shot while moving within it, under the action of the gases generated by the ignited gunpowder. In discussing briefly this example, we will put out of the question the weight of the gunpowder. The pressures exerted by the gases will neutralise each other, except those exerted on the shot and in the opposite direction on the bottom of the chamber; the former force communicates to the shot its initial velocity, the latter to the gun its velocity of recoil. It must be observed that the force which acts on the shot during its passage along the bore is by no means constant, but decreases rapidly as the gases expand, and the shot approaches the mouth of the gun; at each instant, however, the force on the shot must be equal, and opposite to that on the end of the gun. Consequently all the conditions of the last article meet in this case, and the conclusions therein arrived at are applicable to it.

Ex. 127.—Suppose, for instance, that the gun weighs 12 tons, and the shot 300 lbs., that the length of the bore is 12 ft., and the initial velocity of the shot 1440 ft. a second; we can draw the following conclusions :—

(*a*) The velocity of the gun's recoil is $300 \times 1440 \div 26{,}860$, or 16·07 ft. a second.

(*b*) The energy of the shot and gun are $\frac{1}{2} \times 300 \times (1440)^2$ and $\frac{1}{2} \times 26{,}880 \times (16{\cdot}07)^2$, or 311,040,000 and 3,471,430 units of work respectively.[1]

[1] i.e. 9,662,300 and 107,840 foot-pounds.

(*c*) The recoil would lift the gun through about 4 ft. vertically.

If now we make the further supposition that the force producing motion was constant, we may conclude:—

(*d*) That the force producing motion in the shot or gun amounted to 25,920,000 absolute units, or about 360 tons.

(*e*) That under the action of this constant force, the shot would have described the length of the bore in the 1–60th part of a second.

All these conclusions are easily arrived at, and should be carefully verified by the student. He must not fail to notice, however, that the force determined in (*d*) is not that which actually comes into play, but its mean value. Also with regard to the conclusion (*e*), it must be noted that if the shot described the length of the bore with the velocity it has on leaving the mouth of the gun, it would do so in the 1–120th of a second. Now, as the shot begins to move from rest, and acquires the velocity of 1440 ft. a second at the end of the bore, the actual time of its motion within the gun must exceed the 1–120th of a second. On the other hand, as the motion takes place under a decreasing force, the time of its motion within the gun will be less than 1–60th of a second. The point, however, which the conclusions (*d*) and (*e*) are intended to bring out is not the exact numerical values of the force and time, but the facts that the motion of the shot is caused by a very great, though finite force, and that this force acts for a finite, and indeed measurable, though very short, interval of time.

110. *Impact.*—When a blow is struck by one body against another, a mutual action is caused which gives rise to a case closely resembling that discussed in the last article. If we suppose the body receiving the blow to be fixed, what takes place is as follows:—The bodies in contact compress each other, and the resistance offered by the materials to change of form destroys the momentum of the impinging body; the mutual force exerted during this part of the action is called the force of compression. Afterwards the bodies, while in contact, recover their shape more or less perfectly, and the force exerted during this portion of the motion is called the force of restitution. It is this force, due to the elasticity of the materials, which gives rise to the rebound of the impinging body, and to the vibratory motions both of the fixed body on which the blow is struck and of the rebounding body. Some substances have but little tendency to recover their shapes, and with

them the action ceases, or nearly so, with the compression; such substances are clay, putty, &c. With other substances the force exerted during restitution is nearly equal to that called into play during compression; such are ivory, glass, &c. And, of course, there are other substances with corresponding properties intermediate to the two extremes, such as compressed wool, iron, &c.

It is commonly, but most erroneously, supposed that the blow is struck instantaneously; in reality the impact lasts for a very short but finite time, and calls into play a finite force which increases from zero up to a very large magnitude at the instant of greatest compression, and then decreases again down to zero. In illustration of this fact we will consider the following case:—

Ex. 128.—A ball weighing 10 lbs. falls from a height of 9 ft. on to a floor which yields through 1–12th of an inch. Supposing the ball to be absolutely incompressible, determine the mean value of the force called into play during the compression of the floor; and supposing that the actually acting force had been a force with this mean value, determine the time during which it must have acted. Here, the energy of the ball at the instant of contact is $9 \times 10\,g$ units of work; this number of units of work is, therefore, done against a force (R) acting through 1–12th of an inch, or 1–144th of a foot. Hence, assuming R to be a constant force, we have $9 \times 10\,g = \frac{1}{144} \times R$, and therefore R is a force of $12{,}960\,g$ absolute units, or nearly 6 tons. The velocity with which the ball strikes the floor is 24 ft. a second (Art. 89), and it loses this velocity while describing 1–144th of a foot, and therefore, supposing the resistance to be a constant force, the time will be $\frac{1}{144} \div 12$, or the 1–1728th of a second. If the body is supposed to rebound to a height of 4 ft., the student will easily show that the mean value of the force of restitution is $5760\ g$ absolute units, or about $2\frac{1}{2}$ tons, and would act—were it a constant force—for 1–1152nd of a second, supposing the contact to cease at the instant the floor has just recovered its shape.

The student will notice that these results are obtained by making a number of suppositions, which would not exactly hold good in reality; still the calculation shows very distinctly the sort of action which really takes place, viz. a mutual action rapidly increasing from zero up to a very large force, and then rapidly decreasing, the mean value during the increase being 6 tons, and during the decrease $2\frac{1}{2}$ tons, the whole time being something less than that given by the above supposition, viz. 1–700th of a second. To investigate such a question as that given above, taking account of all the

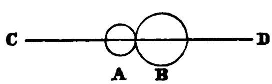
conditions of the case, is not easy; it results, however, from such an investigation that, if two balls of the dimensions of the earth, and of some such degree of rigidity as copper, steel, or glass, were to come into direct contact with unequal velocities, the whole process of compression and restitution would occupy a time not differing much from an hour. In the case, however, of globes of these substances not exceeding a yard in diameter, the whole process is probably over within a thousandth of a second.[1]

111. *Remark.*—The last article puts in a clear light a point which the student should carefully bear in mind: a force may act upon a body for so short a time that the body scarcely changes its position while the force acts, and yet during the same time acquires under the action of the force a finite, and even a considerable, velocity. This is, of course, an immediate consequence of the fact that the velocity acquired is proportional to the time, while the space described is proportional to the square of the time (Art. 89). Thus, as we saw in the last article, the force of restitution acted for the 1–1152nd of a second, and that during this time the body moved through only 1–12th of an inch, but acquired a velocity sufficient to raise it to a height of 4 ft., i.e. it acquired a velocity of 16 ft. a second, or more than 10 miles an hour. In further illustration of this point the student may consider the following case:—

Ex. 129.—A force of 100,000 absolute units acts on a body weighing 1 lb. for the 1–10,000th of a second; it is required to find the distance described and the velocity acquired by the point while under the action of the force. Using the formula $\text{M}f=\text{P}$, we see that $f=100,000$; consequently by Art. 89, $s=$ 1–2000th of a foot, and $v=10$ ft. a second.

112. *Direct impact.*—The question to be considered under this head is as follows:—A point, or smooth ball, whose mass is A, moving without rotation with a velocity U along the line C D, overtakes another point, whose mass is B, moving along C D with a velocity V; required the velocities (u and v) of A and B immediately after impact. For the solution of this question it is not necessary to know the exact magnitude of the forces at each instant of the time during which the impact lasts. What is necessary, and what we know to be the case, is that whatever be the force with which A acts on B, urging it towards D, an

Fig. 100.



[1] *Treatise on Natural Philosophy*, by Thompson and Tait, p. 206.

exactly equal opposite reaction is exerted by B against A. We know, therefore (Art. 108), that whatever momentum A communicates to B, an equal momentum, but in the opposite direction, is communicated by B to A; in other words, the momentum gained by B must equal that lost by A, and this is true for the whole or any part of the time during which the mutual action lasts. Moreover, it is plain that the compression of the bodies will last as long as A has a greater velocity than B, and will cease as soon as the velocities are equalised. Consequently if x is the common velocity of the bodies at the end of compression, the momenta are respectively $\mathrm{A}x$ and $\mathrm{B}x$ at that instant. But if R is the momentum lost by A and gained by B during compression, the momenta are respectively $\mathrm{AU}-\mathrm{R}$ and $\mathrm{BV}+\mathrm{R}$. Consequently we have these two relations—

$$\mathrm{A}x = \mathrm{AU} - \mathrm{R} \qquad (1)$$

and

$$\mathrm{B}x = \mathrm{BV} + \mathrm{R}. \qquad (2)$$

These equations determine R and x; and if the mutual action ceases with compression, the question is solved, the case in which there is no restitution being one extreme case. If we suppose there is restitution, the mutual action will be continued, and consequently A will experience a further loss and B a further gain of momentum. If we take the other extreme case, and suppose the restitution perfect, the change of momentum during restitution is the same as the change during compression; consequently during the whole impact A loses and B gains a momentum 2 R. If therefore u and v are respectively the velocities of A and B at the end of the impact, we have

$$\mathrm{A}u = \mathrm{AU} - 2\mathrm{R} \qquad (3)$$

and

$$\mathrm{B}v = \mathrm{BV} + 2\mathrm{R}. \qquad (4)$$

As R has already been determined, these equations give u

and v. The student will have no difficulty in writing down these equations if he understand the principles on which they depend. In applying them to any particular case, he must reckon those velocities positive which are in one direction (say left to right), and those negative which are in the opposite direction (right to left). The equations will then give results free from ambiguity, whether the one body overtakes the other or whether they meet.

Ex. 130.—Let the masses of A and B be respectively 3 and 5, and their velocities 24 and 16; required their velocities after impact—(1) Supposing the action to cease with compression; (2) supposing the restitution complete.

In the former case we have $A = 3$, $B = 5$, $U = 24$, $V = 16$; consequently Equations (1) and (2) become

$$3x = 72 - R,$$

$$5x = 80 + R.$$

Whence $$x = 19, \text{ and } R = 15,$$

i.e. the common velocity at the end of compression is 19, and the momentum gained by the one body and lost by the other is 15.

In the latter case the total momentum interchanged would be 30, and therefore the velocities (u and v) with which the bodies separate are

$$3u = 72 - 30$$

and $$5v = 80 + 30;$$

i.e. A moves on with a velocity of 14, and B with a velocity of 22.

Ex. 131.—If in the last example the mass of B had been 21, and its velocity 6, the common velocity at the end of compression would have been $8\frac{1}{4}$, and the momentum interchanged $47\frac{1}{4}$; consequently on the supposition of the restitution being perfect, the whole interchange of momentum is $94\frac{1}{2}$, and therefore u equals $-7\frac{1}{2}$, and v equals $10\frac{1}{2}$; in other words, A rebounds with a velocity $7\frac{1}{2}$, and B moves on with a velocity $10\frac{1}{2}$.

Ex. 132.—Let the masses of A and B be 3 and 5, and suppose that A moving from C to D with a velocity 24, meets B moving from D to C with a velocity 16. Required the velocities after impact.

Here $U = 24$ and $V = -16$; consequently Equations (1) and (2) give

$$3x = 72 - R,$$

$$5x = -80 + R.$$

Whence $x = -1$ and $R = 75$, i.e. if the materials were such that the action ceased at the end of compression, the bodies would both be moving in the

direction D to C, with a velocity 1. If we suppose, however, that the restitution is complete, the momentum of each body will undergo a change 2 R or 150, and therefore we obtain from Equations (3) and (4)

$$3u = 72 - 150,$$
$$5v = -80 + 150.$$

Hence $u = -26$ and $v = +14$, i. e. A moves back towards C with a velocity 26, and B moves back towards D with a velocity 14.

113. *Direct impact against a fixed body.*—In this case, at the end of compression the impinging body must be brought to rest; consequently Equation (2) of the last article gives

$$0 = \text{B}\,\text{V} + \text{R};$$

i. e. $\text{R} = -\text{B}\,\text{V}$. If now we suppose the restitution perfect, the whole change of momentum will be $-2\,\text{B}\,\text{V}$, and therefore Equation (4) becomes

$$\text{B}\,v = \text{B}\,\text{V} - 2\,\text{B}\,\text{V};$$

i, e. $v = -\text{V}$; in other words, the body rebounds with a velocity equal to that with which it strikes the plane. That bodies never rebound with a velocity equal to that of impact, is due to the fact that their mutual action ceases before the restitution of form is complete.

114. *Change of energy in impact.*—The question whether the whole energy of the bodies is changed by impact is easily answered when the equations of Art. 112 have been solved. For, at the instant before impact, the total energy of the bodies is $\frac{1}{2}(\text{A}\,\text{U}^2 + \text{B}\,\text{V}^2)$; at the end of compression it is $\frac{1}{2}(\text{A}\,x^2 + \text{B}\,x^2)$, and at the end of the restitution it is $\frac{1}{2}(\text{A}\,u^2 + \text{B}\,v^2)$. Thus, in Ex. 132 we find that before impact the energy is 1504, at the end of compression 4, at the end of restitution 1504. The student will observe that in this case the total energy at the end of the impact is the same as at the beginning, and he will easily prove that this is always true when the restitution is perfect. If, however, the restitution is not perfect, the total energy at the end is less than the total energy at the beginning of the impact. The question may be asked, What has become of the remainder?

Before giving an answer to this question, it will be convenient to notice a distinction between two kinds of energy—one kinetic energy or energy of motion, the other potential energy. Kinetic energy is the same as what we have hitherto termed energy, or accumulated work, or vis viva. The meaning of the term 'potential energy' will be best explained, for our present purpose, by an example or two. Suppose a mass of 10 lbs. is raised to a height of 5 ft.; to produce this result 50 foot-pounds of work must have been done against gravity. If, however, the body were allowed to fall to the place it came from, it would do against a resistance 50 foot-pounds of work; its capacity for doing this work depends on its position, and is called the 'potential energy' of the body, as due to its change of position.

Suppose a spring brought into a state of compression by an expenditure of 50 foot-pounds of work, if it were allowed to recover its shape, it would do 50 foot-pounds of work against a resistance. The power which the spring has of doing work, in consequence of its state of compression, is its 'potential energy' due to that state. Suppose work to be expended in rubbing two rough bodies together, their temperature will be raised, and on the supposition that none of the heat thus generated is dissipated, the bodies, in the act of cooling down to their original temperature, could do just as many units of work as were expended in raising the temperature; the additional temperature imparted to the bodies, so long as it is retained, is therefore another form of 'potential energy.' We are now in a position to answer the question what had become of the energy in Ex. 132, when it had diminished from 1504 units at the instant of impact to 4 units at the end of compression. The answer is, that the 1500 units of kinetic energy had been momentarily converted into potential energy, being replaced by the state of compression existing at that instant. During the restitution of the shapes of the bodies, the potential energy is reconverted into kinetic energy, and if the mutual action continues till the restitution is perfect (the case contemplated in Ex. 132), the whole amount of kinetic energy at the end of the impact is the same as at the beginning of the impact. If, however, the bodies cease to be in contact before the restitution is complete—and this is what usually happens—only part of the potential energy is reconverted into the kinetic energy of the translatory motion of the bodies; the remainder continues in the bodies in the form of vibrations, and is gradually replaced by its equivalent of heat in consequence of internal friction.

QUESTIONS.

1. State exactly what are the three primary units of distance, time, and mass (Art. 84 and 2).

2. What is meant by uniform velocity? How is it measured?

3. A body moves with a velocity 8; how far will it go in half an hour? *Ans.* 4800 yds.

4. A body moves at the rate of 50 miles per $1\frac{1}{2}$ hour; what is its velocity in feet per second? *Ans.* $48\frac{8}{9}$.

5. The equatorial diameter of the earth is 41,847,000 ft.; the earth makes one revolution in 86,164 sec.; what is the velocity, due to rotation, of a point on the equator? *Ans.* 1526 ft. a second.

6. State how variable velocity is estimated, and give instances.

7. What is meant by a uniformly accelerated velocity? A body is moving at a given instant at the rate of 5 ft. a second; at the end of 4 sec.

its velocity is 23, at the end of 6 sec. its velocity is 32; is this consistent with uniform acceleration?

8. A body is moving at a given instant at the rate of 8 ft. a second; at the end of 5 sec. its velocity is 19. Assuming its velocity to undergo uniform acceleration, what was its velocity at the end of 4 sec., and what will be its velocity at the end of 10 sec.? *Ans.* 16·8; 30.

9. A body moving at any instant with a velocity of 30 miles an hour, subject to a uniform retardation, comes to rest in 11 sec.; what was its velocity 5 sec. before it stopped? *Ans.* 20.

10. A body moves at the rate of 12 ft. a second, its velocity undergoes a uniform acceleration 4: (1) State exactly what is meant by the number 4; (2) suppose the acceleration to go on for 5 sec., and then to cease, what distance will the body describe between the ends of the 5th and 12th sec.? *Ans.* 224 ft.

11. State the reasoning by which it can be shown that when a body begins to move with a velocity v, which undergoes a uniform acceleration f, the distance described in a time t is given by the equation $s = \mathrm{v}\,t + \frac{1}{2} f t^2$. Also state the reasoning adapted to the case of a body beginning to move from rest, and to that in which the velocity is uniformly retarded. If t is estimated in seconds, and s in feet, state exactly how v and f must be estimated (viz. v in feet per second, and f is the additional velocity in *feet* per *second*, acquired in *each second* of the motion).

12. A body moves from rest, its velocity undergoing a uniform acceleration 8; find the distances it describes in each of the first 4 sec. of its motion, and show that they are in arithmetical progression.

13. Show that in uniformly accelerated or retarded motion the distances described in successive equal times are in arithmetical progression.

14. A body, whose velocity undergoes a uniform retardation 8, decribes in 2 sec. a distance of 30 ft.; what was its initial velocity? For how much longer than the 2 sec. would it move before coming to rest? *Ans.* (1) 23; (2) $\frac{7}{8}$ sec.

15. A body, whose motion is uniformly retarded, changes its velocity from 24 to 6, while describing a distance of 12 ft.; in what time does it describe the 12 ft.? *Ans.* 0·8 sec.

16. A body undergoing a uniform retardation 32, begins to move with a velocity of 24 ft. a second; how far will it move before coming to rest? *Ans.* 9 ft.

17. The velocity of a body, which is at first 6 ft. a second, undergoes a uniform acceleration 3; at the end of 4 sec. the acceleration ceases; how far does the body move in 10 sec. from the beginning of the motion? *Ans.* 156 ft.

18. A body moves for a quarter of an hour with a uniformly accelerated

velocity; in the 1st five minutes it describes 350 yds., in the 2nd five minutes 420 yds.; what is the whole distance described in a quarter of an hour? *Ans.* 1260 yds.

19. Give fully the reasoning by which it can be shown that an acceleration 220 reckoned in yards and minutes is the same as one of 450 in miles and hours, and as one of $\frac{11}{60}$ in feet and seconds.

20. In Q. 18 express the initial velocity and acceleration in feet and seconds; and with these units find the space described in the quarter of an hour. *Ans.* (1) 3·15; (2) $\frac{7}{3000}$.

21. What is meant by the accelerative effect of a force? Illustrate the answer with reference to the accelerative effect of gravity near the earth's surface. What may be taken as the numerical value of this effect unless great exactness is necessary? What is meant by a retarding force? Give an example.

22. If it were found that a body falling from a state of rest under the action of gravity described 16 ft. in the 1st second, 48 in the 2nd second, 80 ft. in the 3rd second, and so on, what conclusion could be drawn as to the accelerative effect of gravity?

23. A body is thrown upward in vacuo, with a velocity of 112 ft. a second; after how many seconds will it return to the starting point? *Ans.* 7 sec.

24. A and B are two points in a vertical line, A being 72 ft. above B; a body is thrown downward from A with a velocity of 27 ft. a second; at the same instant another body is let fall from B; after how long will the former body overtake the latter? *Ans.* $2\frac{2}{3}$ sec.

25. What is the momentum of a body? If a force acting on a mass m communicates in each second an additional velocity of f ft. a second, what momentum does the force communicate to the body in each second? State three fundamental facts regarding the effect of a force acting on a body (Art. 92).

26. Two bodies, whose masses are 49 lbs. and 2 cwt., move with velocities of 16 ft. a second and 210 yds. a minute respectively; compare their momenta. *Ans.* 1 : 3.

27. The mass of a body (A) is five times that of another body (B); a force acting on A communicates to it in 1 sec. a velocity of 4 ft. a second; another force acting on B communicates to it at the end of 3 sec. a velocity of 36 ft. a second; in what ratio are these forces? *Ans.* 5 : 3.

28. A certain force acting on a mass of 1 cwt. makes it describe from rest a distance of 10 ft. in 1 sec.; what velocity would an equal force communicate in 3 sec. to a mass of 1 ton? *Ans.* 3 ft. a second.

29. Two masses of 7 and 5 lbs. respectively are observed to begin to move

from rest, and to continue side by side during their motion; what conclusion can be drawn as to the forces acting on the bodies?

30. What is meant by the absolute unit of force?

31. A mass of 15 lbs. acquires in 4 sec. a velocity of 24 ft. a second; how many absolute units were there in the force producing this velocity? If an equal force acted on a mass of 50 lbs. for 7 sec., what distance would it describe from its position of rest? *Ans.* (1) 90 units; (2) 44·1 ft.

32. Determine the number of absolute units of force in a gravitation unit. If g is the accelerative effect of gravity at any place, how many absolute units of force does gravity exert on M lbs. of matter at that place?

33. If, at a place where g equals 32, 12 lbs. of matter will compress a spring through a certain distance, how many pounds of matter would be required to compress the same spring through an equal distance at a place where g is 18? *Ans.* $21\frac{1}{3}$.

34. Assume for simplicity that g equals 32 on the earth's surface; consider Ex. 97 and 98, and answer this question:—If the actions took place on a planet, where the accelerative effect of gravity is 20, what would now be the answers to the questions? *Ans.* (1) 63,360 gals.; (2) 24 bushels.

35. Why would a hypothesis similar to that in the last question make no change in the results of Q. 36, Chap. IV.?

36. What is the relation between mass, force, and acceleration, when the mass moves on a smooth horizontal plane?

37. A force of 3 gravitation units (say 96 absolute units) acts horizontally on a mass of 108 lbs., placed on a smooth horizontal plane; how far will it make the body move from its position of rest in 5 sec.?

Ans. $11\frac{1}{9}$ ft.

38. What is the effect of supposing the plane in Q. 36 to be rough?

39. A mass is moving at the rate of 12 ft. a second on a rough horizontal plane ($\mu = 0{\cdot}18$), how far will it move before coming to rest? Why is the answer to this question independent of the mass of the body?

Ans. 12·5 ft.

40. A body slides up or down an inclined plane; find the acceleration (or retardation) of its velocity, (1) when the plane is smooth, (2) when the plane is rough.

41. The inclination of a plane is 3 vertical to 4 horizontal; a body is made to slide up the incline with an initial velocity of 36 ft. a second; assuming the plane to be smooth, how far will it go before beginning to return, and after how many seconds will it return to its starting point?

Ans. (1) $33\frac{3}{4}$ ft.; (2) $3\frac{3}{4}$ sec.

42. In the last question, if the plane were rough, and the coefficient of friction were $\frac{1}{4}$, find the time it takes the body to return to its starting point, and the distance it describes. *Ans.* (1) 3·4 sec.; (2) $50\frac{5}{8}$ ft.

43. There is a smooth inclined plane of 5 vertical to 12 horizontal: a body slides down 52 ft. of its length, and then passes without loss of velocity on to the horizontal plane, which is also smooth; after how long from the beginning of the motion will it be at a distance of 100 ft. from the foot of the incline? *Ans.* 5·7 sec.

44. In the last case suppose the plane rough, the coefficient of friction being $\frac{1}{4}$; find where the body comes to rest, and the time of the motion. *Ans.* (1) 32 ft. from foot of plane; (2) 7·42 sec.

45. Determine the acceleration of the motion (1) when P by falling raises Q, P and Q being bodies connected by a fine thread passing over a smooth block; (2) when P by falling draws Q along a horizontal plane (rough or smooth), the bodies being connected as before.

46. If, in Art. 99, Q is a mass of 20 lbs., and is found to ascend from a state of rest through 5 ft. in the 1st three seconds of the motion, what is the mass of P? How high would Q ascend in the 4th second of the motion? *Ans.* (1) $21\frac{61}{139}$ lbs.; (2) $3\frac{8}{9}$ ft.

47. In Ex. 117, if P comes to the ground after falling through 3 ft., how far will Q move along the table before coming to rest? *Ans.* $\frac{5}{8}$ ft.

48. In Ex. 117, suppose the table smooth, and that at the beginning of the motion P is 4 ft. above the ground, and Q 12 ft. from the edge of the table; find how many seconds it will take P to reach the ground, and Q the edge of the table. *Ans.* (1) 1·225 sec.; (2) 2·45 sec.

49. What two principal difficulties are encountered in verifying the laws of falling bodies? Give an example of the resistance offered by the air to the motion of falling bodies.

50. What is meant by two intervals of time being equal? How, practically, may successive intervals of time be ascertained to be equal?

51. Describe an experiment from which it can be inferred that at a given place the force of gravity on different bodies is proportional to their masses.

52. Describe an experiment of motion on an inclined plane, from which it can be inferred that the accelerative effect of gravity on bodies at a given place is constant.

53. Describe Atwood's machine. How can the machine be used to show (1) that, with given masses and over-weight, the acceleration is constant, (2) that forces acting on equal masses are proportional to their accelerative effects (3) that, generally, forces are proportional to the momenta they generate in a given time?

54. In Atwood's machine the boxes weigh 10 oz. each, the over-weight is $\frac{1}{2}$ oz., and it is found that after falling for $2\frac{1}{2}$ sec. to the ring, the body takes 3 sec. in falling to the stage, which is 5 ft. below the ring; show that the inertia of the pulley is equivalent to $3\frac{1}{2}$ oz., i. e. the motion is the same as if the pulley were without inertia, and the mass of each box increased by $1\frac{3}{4}$ oz.

55. In the last question, if the over-weight were increased to $1\frac{1}{2}$ oz., what velocity would it impart to the weights in $2\frac{1}{2}$ sec.?

Ans. 4·8 ft. a second.

56. Describe Morin's machine, and explain the contrivance by which the time of the motion of the falling body is ascertained.

57. When it has been ascertained by experiment that gravity has a constant accelerative effect on the motion of a falling body, why is there no need of experimental proof that the equations of Art. 89 apply to bodies falling in vacuo?

58. State briefly the evidence for the statement that the coefficient of friction, in the case of a body moving on a rough plane, is independent of the velocity.

59. If a mass of M lbs. is moving at any instant with a velocity of V ft. per second, how does it appear that its energy is equivalent to $\frac{1}{2}$ M V^2 units of work? What unit of work is intended in this question? What expression would give the energy of the body in foot-pounds?

60. A body weighing 4471 lbs. is moving at the rate of 5 ft. a second; what is its energy in foot-pounds (1) exactly, (2) approximately?

Ans. (1) 1736; (2) 1746.

61. If the body in the last question were brought to rest by a constant force of 450 absolute units, through what distance must the force act?

Ans. 124·2 ft.

62. If a point, whose mass is M, moves with a velocity V, what assertions can we make respecting the work done by the forces which communicated the velocity, and respecting the work which will be done against the resistances which bring the mass to rest? Hence, explain why the quantity $\frac{1}{2}$ M V^2 is sometimes called the accumulated work of the point.

63. A body weighing 1 lb. falls from a height of 20 ft., and is brought to rest by a constant force acting through 1 in.; find that force, (1) if g equals 32, (2) if g were to equal 24.

Ans. (1) 7680 abs. un.; (2) 5760 abs. un.

64. A train weighing 100 tons, exposed to resistances at the rate of 8 lbs. a ton, is driven with a uniform velocity by an engine of 64 h.-p.; what will be its velocity and energy? If it is required to bring the train to rest

in $\frac{1}{3}$ mile, by turning off the steam and putting on the break, to what must the resistances be increased?

Ans. (1) 30 miles an hour; (2) 6,776,000 foot-pounds;
(3) 38·5 lbs. a ton.

65. A body whose mass is 1 lb. slides down a rough inclined plane, whose height is 9 ft. and base 12 ft.; find its energy at the foot of the plane, and how far that energy would carry it along the horizontal plane; the coefficient of friction throughout being $\frac{1}{8}$, and no velocity being lost in passing from the inclined to the horizontal plane. *Ans.* (1) 240 units; (2) 60 ft.

66. In the last question, if g were supposed to equal 24, why would this affect the first answer and not the second?

67. When the velocity of a point whose mass is M changes from V to v, show that the change in its energy is $\frac{1}{2}$ M $(v^2 - V^2)$ units of work.

68. When a body is brought to rest by a constant force, show that, while losing the first half of its energy, it describes three times the distance that it describes in losing the second half.

69. When a machine moves for a given time, what relation exists between the work done by the power, the work done against the weight and the passive resistances, and the energy of the parts of the machine?

70. Draw a triangle ABC, right angled at C; a mass P (of 50 lbs.) is at A, at the foot of the plane; it is fastened by a thread, which passes over a smooth point at B to a mass Q (of 20 lbs.); given that the base AC is 240 ft., and the height BC 7 ft., that Q is allowed to fall, and to draw P up the plane; find the velocity acquired by P on reaching B.

Ans. 63·8 ft. a second.

71. In the last question, if the coefficient of friction between P and the plane were 0·1, what would P's velocity be on reaching B?

Ans. 54·5 ft. a second.

72. Determine the time in which P is drawn from A to B, in the last case. *Ans.* 8·8 sec.

73. A mass of 50 lbs. is placed in a basket attached by a rope to a drum; it is found that the weight causes the basket to descend through 3, 9, 15, . . . feet in the 1st, 2nd, 3rd . . . second of its motion; what is the reaction of the basket against the mass? *Ans.* 1300 abs. un.

74. In Art. 100, suppose P and Q to be masses of 30 and 70 lbs., and that the coefficient of friction between Q and the table is 0·2; find the amount of the action of P on Q. *Ans.* 806·4 abs. un.

75. In Ex. 126, suppose the table to be rough, and the coefficient of friction to be 0·1. Find (1) the mutual action between the bodies; (2) the addition that must be made to P to keep the acceleration the same as in

Ex. 126; (3) the mutual action between the bodies in these altered circumstances. *Ans.* (1) $586\frac{2}{3}$ abs. un.; (2) 12 lbs.; (3) $853\frac{1}{3}$ abs. un.

76. What is meant by the force of inertia of a body? On what is the force of inertia of a body exerted?

77. In Q. 74 how much of the force transmitted along the thread to P is due to Q's inertia? *Ans.* 358·4 abs. un.

78. In Q. 70 find how much of the action of P on Q is due to P's inertia. *Ans.* 423·8 abs. un.

79. When two bodies are known to be acted on for equal times by equal forces, what conclusions can be drawn as to the momenta, and as to the energies of the bodies due to the action of the forces? Why is it not necessary that the forces should be constant?

80. Describe briefly the action of the gases produced by the ignition of gunpowder on the shot while moving along the bore of the gun, and on the gun itself.

81. A shot weighs 1 oz., and has an initial velocity of 1200 ft. a second, the gun weighing 20 lbs., and having a barrel 4 ft. long; find (1) the velocity of the recoil, (2) the energy of shot and gun, (3) the average amount of the mutual pressure between shot and gun, (4) the time in which the shot, if acted on by this average pressure, would move along the bore. *Ans.* (1) $3\frac{3}{4}$ ft. per sec.; (2) shot, 45,000 units; gun, $140\frac{5}{8}$ units; (3) 11,250 abs. un.; (4) $\frac{1}{150}$ sec.

82. Describe the mutual action which takes place in the direct impact of two bodies.

83. It is required to impart a velocity of 44 ft. a second to a mass of 50 lbs. by a force which during its action causes the body to move through 1 in.; find (1) the magnitude of the force, (2) the time of its action, (3) the mass of the body which such a force would sustain by direct opposition against the action of gravity in London.

Ans. (1) 580,800 abs. un.; (2) $\frac{1}{264}$ sec.; (3) 18,042 lbs.

84. When one body impinges directly on another, explain how to form the equations which determine the interchange of momentum up to the end of compression, and the common velocity of the bodies at that instant.

85. If the restitution is complete, explain how to find the velocities of the bodies at the end of impact in the last question.

86. How can the change in the joint energy of two bodies after impact be determined?

87. A body (A) weighing 5 lbs., and moving with a velocity 15, meets a body (B) weighing 10 lbs., and moving with a velocity 4; find (1) their

common velocity at the end of compression, (2) their velocities at the end of restitution supposed to be perfect.

Ans. (1) $2\frac{1}{3}$ in the direction of A's motion; (2) both rebound, A with a velocity $10\frac{1}{3}$, B with a velocity $8\frac{2}{3}$.

88. A mass of 100 lbs., perfectly free to move, is struck by a body weighing 2 lbs., and moving with a velocity of 1000 ft. per second, which penetrates the mass and remains in it:—(1) Why must this case be treated as one in which the mutual action ceases at the end of compression? (2) What is the velocity with which the mass begins to move? (3) What is the energy of the mass at the end of the impact? (4) If the body were on a rough horizontal plane, how far would it move before coming to rest, the coefficient of friction being 0·2?

Ans. (2) $19\frac{31}{51}$ ft. per second; (3) 19,608 units; (4) 30 ft.

89.—Give examples to illustrate the meaning of the term 'potential energy.' In the last question, how much of the energy of the shot has been converted into 'potential energy' at the end of the impact?

Ans. 980,392 units.

N.B.—In the answers to the above questions, and elsewhere in this book, 32 is taken as the numerical value of the accelerative effect of gravity, unless the contrary is expressly stated or manifestly implied; e.g., in obtaining the 3rd answer to Q. 83 g is taken to equal 32·1912, this being manifestly implied in the mention of the place.

CHAPTER VI.

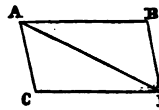
MOTION IN A CIRCLE.

115. *Composition of velocities.*—We have hitherto considered only those cases in which the force acts on the body along the line in which the motion takes place. We are now to consider one or two simple cases in which the force acts along a line transverse to the direction of the motion. In other words, hitherto we have considered only cases in which the force changes the velocity of the motion. We are now to consider some cases in which the force changes the direction of the motion. To enable us to do this we must consider the following question:—

A certain point has two velocities simultaneously impressed on it in different directions; what are the direction and magnitude of the resultant velocity? For instance, a ship sails due north at the rate of 4 miles an hour, a man walks across its deck towards the west at the rate of 4 miles an hour. The man has two velocities simultaneously impressed on him, viz. the velocity of the ship, and that due to his walking, and in consequence he is approaching the north and the west simultaneously with equal velocities, so that his actual motion is to the north-west. A little consideration will show that this result is generally true, so that if the one velocity is represented in magnitude and direction by A B, the other in magnitude and direction by A C, the resultant velocity will be represented in magnitude and direction by the diagonal A D of

the parallelogram, whose sides are A B and A C. On the other hand, a velocity represented in magnitude and direction by A D is equivalent to two coexistent velocities represented in magnitude and direction by the sides A B and A C of any parallelogram A B D C constructed on A D as a diagonal. Thus, if a train moves in a north-west direction with a velocity of 50 miles an hour, it approaches the north with a velocity a little exceeding 35 miles an hour, and simultaneously approaches the west with an equal velocity. On the whole, it will be seen that velocities admit of composition and resolution by the same rule as forces.

FIG. 101.

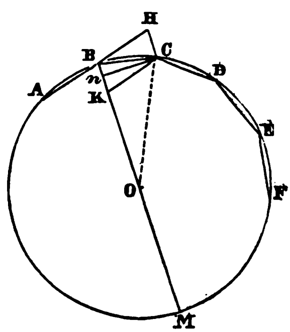


116. *Applications.*—We shall now deduce two results from the last article which will be useful in certain questions which follow.

(1) Let A B C D be any regular polygon inscribed in a circle whose centre is O. If A B is produced to H, so that B H and A B are equal, it is plain that H C is parallel to B O; if, then, K C is drawn parallel to B H, B H C K is a parallelogram whose diagonal B C is equal to the side B H. Now suppose a body to move from A with a given velocity (V), and when it reaches B that another velocity (v) is impressed upon it in the direction B O, such that

$$v : V :: BK : BH.$$

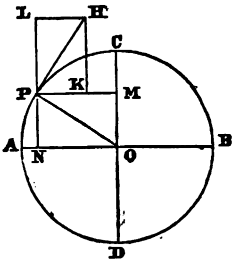
FIG. 102.



At the point B the body has simultaneously two velocities v and V, represented in magnitude and direction by B K and B H; consequently the resulting velocity is represented in magnitude and direction by B C; in other words, the consequence of the impressed velocity (v) is merely to change the direction of the motion, the velocity of the body being still V, but the motion taking place along B C. If when the body reaches C a velocity equal to v is again impressed on it along the line C O, the body will describe the side C D with the velocity V. In a like manner it may be made to describe in succession all the sides of the polygon with the constant velocity V.

(2) Suppose a body to move with a uniform velocity in a circle ACBD. Through the centre O draw two diameters AB, CD at right angles to each other. It is required to determine the components of the velocity in directions parallel to AB and CD. Take P any point in the circumference; join P and O; draw PN and PM at right angles to AB and CD respectively. Draw PH at right angles to PO, and equal to it; complete the rectangle PKHL; PL is in NP produced, and it is plain that PK is equal to PN, and PL to PM. Now as PH touches the circle at P, the body when at P is moving in the direction PH; consequently PH may be taken to represent the velocity; and, if this is done, PK and PL represent the component velocities required. We see, therefore, that when the body is at any point P, the velocity parallel to AB is to the velocity in the circle as the perpendicular line PN is to the radius PO.

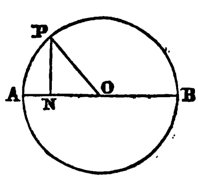
FIG. 103.



A result of some importance can easily be deduced from this. Take AB, any line whose middle point is O, and on AB as a diameter describe a circle; take any point P in the circumference, and draw PN at right angles to AB, and join P and O. Suppose two bodies to set out from A, the one moving along the circumference with a uniform velocity V, the other moving along the diameter with a variable velocity (v), such that at any point (N)

$$v : V :: PN : PO.$$

FIG. 104.



Under these circumstances the bodies will reach B together; for at each instant they are moving with equal velocities in the direction AB. Also since the former body describes the semicircle with a uniform velocity V, the time occupied by the bodies in reaching B will equal $\pi AB \div V$.

117. *Uniform motion in a circle.*—When a point whose mass is M moves with a uniform velocity (V) in a circle whose radius is r, it must be acted on by a force P, given by the equation

$$P = \frac{M V^2}{r}.$$

This force acts along the radius towards the centre. Or, *if more forces* than one act on the body, their resultant

will be equal to the force P, and will act as stated, viz. along the radius and towards the centre. In the above formula, if M is in pounds, r in feet, and V in feet per second, P will be in absolute units.

Ex. 133.—A mass of 12 lbs. is tied to the end of a string 8 ft. long, and is whirled round in a circle at the rate of 10 ft. per second; P will equal $12 \times 10^2 \div 8$, or 150 absolute units, equal to about 4·6875 lbs. And this force must be exerted along the string towards the centre.

118. We will now consider the reason of the statements made in the last article, and, *first*, that the force must always be directed towards the centre. We have already seen (Art. 116) that if a point describes the sides of a regular polygon with a uniform velocity, there must be impressed upon it at each angle a certain velocity in the direction of the radius towards the centre; this velocity must be communicated by a force acting intermittently in the direction of the velocity, i.e. along the radius towards the centre. Now this result is true, however great be the number of sides, and therefore will be true of the limiting case to which the case approaches when the number of sides becomes very great, and the intervals between the successive exertions of the force very small. In this case we have the body moving in a circle with a uniform velocity, and force acts continuously along the radius towards the centre. *Secondly*, suppose the force to be one of P units, then if $P = M f$, the velocity which P will communicate to the body in a time t is $f t$. In fig. 102 draw Cn at right angles to B M; then we know from geometry that

$$BC^2 = BM \times Bn.$$

Suppose the body to describe one side of the polygon in a time t, then $BC = Vt$. Also, if we suppose the velocity $f t$ to be suddenly impressed on the body at B, we have B K,

or 2 B n, equal to $f t \times t$; and as B M is denoted by $2r$, the above equation gives us

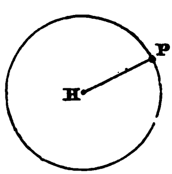
$$V^2 t^2 = \tfrac{1}{2} \cdot f t \times t \times 2r,$$

or $$V^2 = f r.$$

Now this result, being true whatever the value of t, will be true whatever be the length of the side, or whatever the number of the sides of the polygon, and therefore will be true when the body moves in a circle, this being the limit to which the case approaches when the number of sides becomes very great. Now, as we have seen, P equals M f, and consequently it also equals $\frac{M V^2}{r}$, as stated in the last article.

119. *Centrifugal force* is the reaction of the moving body against the fixed point or body whose action constrains it to move in a circle. As the reaction is equal and opposite to the action, the moving body will exert on the fixed point a force equal to M $V^2 \div r$ and along the radius outward. This is a very important point, and one which is frequently misunderstood; we will therefore give the following explanation:—Let P be a mass of 24 pounds tied to the end of a string H P 3 ft. long, and whirled round at the rate of 21 ft. a second. We have here three bodies: the hand or guiding body H, the moving body P, and the string P H connecting the two. Now the action of the hand on P is a force of 3528 absolute units, and is exerted on P in the direction P to H. P in virtue of its inertia reacts on the hand (H) with an equal force of 3528 absolute units in the direction H to P. This reaction of the moving body on the guiding body is the centrifugal force. The third body, the string, trans-

FIG. 105.



mits both these equal opposite forces, and therefore undergoes a *tension* of 3528 absolute units.

We see, then, that if M is the mass of a body moving with a velocity V in a circle whose radius is r, the centrifugal force, or reaction of the moving body on the guiding body at the centre of the circle, is a force of $MV^2 \div r$ absolute units; M being reckoned in pounds, r in feet, and V in feet per second.

120. *Case in which the velocity is variable.*—We have hitherto spoken of a body moving in a circle with a uniform velocity; it must be understood, however, that if, owing to the action of other forces besides that tending to the centre, the velocity of the body varies, the centrifugal force will undergo a corresponding variation, being equal at any instant to $MV^2 \div r$, if V is the velocity at that instant.

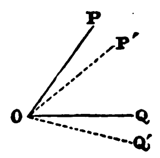
Ex. 134.—Let P be a mass of 24 lbs. moving in a vertical circle, whose radius is 3 ft., and let it pass the lowest point with a velocity of 21 ft. per second. If we suppose the body fastened to the centre by a string, the pull on the centre, transmitted through the string, at this instant will be that due to the centrifugal force of the mass, and the weight of the mass. It will be found that the former force is one of 3528 absolute units; the latter will depend on the force of gravity; if we suppose the accelerative effect of gravity to be 32 (in feet and seconds), it will be 24×32, or 768 absolute units, i.e. the whole pull on the fixed centre is 4296 absolute units. If it were possible for the motion to take place in a situation where the force of gravity is reduced to 25, the pull on the fixed point would be $3528 + 25 \times 24$, or 4128 absolute units. This, of course, is the pull at the instant the body passes the lowest point of the circle.

Ex. 135.—In the last case suppose the body to pass the highest point of the circle with a velocity of 15 ft. a second, what will be the pull on the fixed centre? The centrifugal force is 1800 absolute units; and (assuming that $g = 32$) the force of gravity on the mass is 32×24, or 768 absolute units; consequently the action of the centre on the mass must be a force of $1800 - 768$, or 1032 absolute units; and therefore the reaction of the mass on the centre (i.e. the pull upward on the centre) must be an equal force, viz. one of 1032 absolute units.

121. *Angular velocity.*—Suppose P and Q to be two points of a body turning round an axis which passes

through O at right angles to the plane of the paper. If, in consequence of the rotation, P comes to P′, Q will at the same instant come to Q′, the angle P O P′ being equal to Q O Q′. This follows obviously from the fact that P and Q are two points of a solid body turning round O. The rate at which the angle P O P′ increases is common to the whole body, and is called angular velocity. If we suppose the angle P O P′ to increase uniformly with the time the angular velocity is constant, and is measured by the angle actually described in a second. This angle is estimated in what is called circular measure, i.e. two right angles or 180° are estimated by the number π. Accordingly, if the line O P, joining any point P with O, describes an angle of $n°$ in one second, the angular velocity θ is given by the equation

FIG. 106.



$$\theta = \frac{n\pi}{180}.$$

If the distance O P is denoted by r, the length of the arc described by P in a second is $r\,\theta$, and consequently if v is the velocity of P,

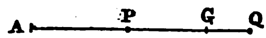
$$v = r\,\theta.$$

If the rotation is not uniform, θ is the rate per second at which the angle P O P′ is increasing at any instant. If m is the mass of the point, its energy is $\frac{1}{2} m v^2$, and its centrifugal force $m v^2 \div r$; but as v equals $r\,\theta$, its energy must be $\frac{1}{2} m r^2 \theta^2$, and its centrifugal force $m r \theta^2$.

Ex. 136.—A body turns uniformly round an axis 20 times a minute; its angular velocity is $20 \times 2\pi \div 60$, or 2·09439; and if we consider a point in it distant 5 ft. from the axis, the velocity of the point will be $5 \times 2{\cdot}09439$, i.e. 10·47195, or very nearly 10·5 ft. a second. If we suppose the point to have a mass of $\frac{1}{3}$rd of a pound, its energy is $\frac{1}{2} \times \frac{1}{3} \times (5 \times 2{\cdot}09439)^2$, or (about) 18·3 units of work (reckoned in feet and absolute units), while the centrifugal force equals $\frac{1}{3} \times 5 \times (2{\cdot}09429)^2$, or 7·3 absolute units.

122. *Centrifugal force of a rotating body.*—Let P and Q be two points whose masses are m and m' respectively, rigidly connected and turning round a centre A, with an angular velocity θ. The centrifugal forces of these points severally are $m \cdot AP \cdot \theta^2$ and $m' \cdot AQ \cdot \theta^2$, and act along the line AQ; consequently the whole centrifugal force is

FIG. 107.



$$(m \cdot AP + m' \cdot AQ)\, \theta^2.$$

Now if G is the centre of gravity of P and Q, we have (Art. 15)

$$(m+m')\, AG = m \cdot AP + m' \cdot AQ,$$

and consequently the centrifugal force is

$$(m+m')\, AG \cdot \theta^2.$$

In other words, the centrifugal force of P and Q is the same as if both masses were placed at G, and continued to revolve with the same angular velocity.

Similar reasoning can be applied to three or more rigidly connected points, and we thus arrive at the conclusion that when any thin plate of matter revolves round an axis at right angles to its plane, the centrifugal force is the same as if the matter were collected at its centre of gravity, and is equal to $M r \theta^2$, where M denotes the mass of the body, r the perpendicular distance of the centre of gravity from the axis, and θ the angular velocity. The same rule is true of a thin plate revolving round an axis in its plane; it also holds good of a solid when symmetrical with reference to two planes both passing through the centre of gravity, one at right angles to and the other containing the axis of revolution, e.g. it is true in the following cases:—(α) A sphere revolving about any axis, (β) a cylinder revolving about a diameter of any circular section, (γ) a cylinder revolving about any axis parallel

to its geometrical axis, (δ) a cone revolving about a diameter of any circular section.

In other cases the rule will hold good with or without modification according to circumstances that need not be defined in this place.

123. *Pressure on the axis of rotation.*—If we suppose the body merely to revolve, and not to be acted on by any external force, the centrifugal force is the whole of the pressure of the body against the bearings of the axis on which it turns. And consequently the pressure on the axis in any of the cases specified in the last article can be easily found.

Ex. 137.—A ball weighing 45 lbs. is fastened to the end of a rod, which turns round an axis at right angles to its length, so that the axis is parallel to a diameter of the ball; the distance from the axis to the centre of the ball is 10 ft.; the number of turns per minute is 80; find the centrifugal force.

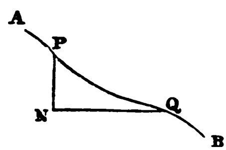
The angle described by the ball in 1 min. is 160π, and therefore the angular velocity is $160\pi \div 60$, or $\frac{8\pi}{3}$. Therefore the centrifugal force is $45 \times 10 \times \left(\frac{8\pi}{3}\right)^2$ or 31,583 absolute units. This force transmitted along the rod would be the whole pressure on he bearings of the axis, if gravity did not act. Since the weight of the ball is 45×32, or 1440 absolute units; if we suppose the circle vertical and the velocity uniform, the pressure on the axis would vary from $31{,}583 + 1440$ absolute units at the lowest point to $31{,}583 - 1440$ absolute units at the highest point of the circle.

124. *Permanent axes of rotation.*—It will be observed that if the axis of rotation passes through the centre of gravity, the pressure on the axis is zero in any of the cases mentioned above, and consequently the body will rotate about such an axis without constraint. Thus, if a plate of any form is caused to rotate about an axis passing at right angles to its plane through its centre of gravity, it will continue to rotate about that axis unless some external force act upon it. The same is true of a sphere rotating about a diameter, of a cylinder rotating about its axis, as is

plain from the last article, and to these several other cases may be added, such as a cone rotating about its axis. In all these cases the axis is said to be one of permanent rotation. It must be remarked, however, that between different permanent axes of rotation there is an important distinction closely resembling the distinction already noticed between stable and unstable equilibrium (Art. 25). For if a body revolving about a permanent axis of rotation is slightly disturbed, the rotation will not (as a rule) be continued round any one axis permanently, but round a number of axes in succession. In some cases these axes will all be very near to the original axis of rotation; in other cases some of the axes will be remote from the original axis of rotation. In the former case the rotation is characterised by stability, being but slightly altered by the disturbance; in the latter case it is characterised by instability, being (though gradually, yet) at last greatly altered by the disturbance.

125. *Variation of velocity of a body describing a smooth curve.*—Let A B be any curve down which a body moves under the action of gravity. Suppose it to have a certain velocity (V) when at P, and another velocity (v) when at Q. Draw the vertical line P N, and let it be cut in N by a horizontal line drawn through Q. The work done by gravity while the body moves from P to Q is M $g \times$ P N, M denoting the mass of the body. As the reaction of the curve does no work (Art. 71), the effect of gravity is simply to increase the energy of the body. But at P the energy was $\frac{1}{2}$ M V^2, and at Q it was $\frac{1}{2}$ M v^2;

FIG. 108.



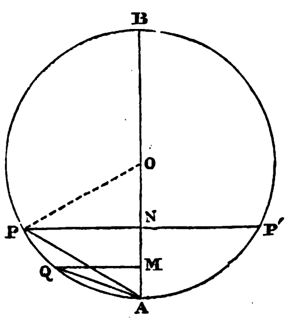
therefore, $$\tfrac{1}{2} M v^2 = \tfrac{1}{2} M V^2 + M g \,.\, PN,$$

or $$v^2 = V^2 + 2 g h,$$

where h denotes the vertical height of P above Q. We

see from this that the change of velocity is the same as would be produced in a body falling vertically from P to N. It may be well to caution the student on two points closely connected with this: (*a*) The *time* in which the body would fall from P to N is very different from that in which the body would move from P to Q; for plainly PQ is much longer than PN, and the velocities of the former body at successive points of PN are the same as at the corresponding points of PQ. (*b*) The formula above proved for the velocity is not true if the curve is rough, since part of the work $Mg \cdot PN$ is done against friction, and only the remainder goes to increase the energy of the point. If, however, the body is attached to a thread, and made to move in a vertical circle, the formula is true; for the constraining force from the centre acts along the thread at right angles to the direction of the motion, and therefore (like the reaction of a smooth curve) does no work.

FIG. 109.



126. *Body moving in a vertical circle.*—Let AB be the vertical diameter of the circle, and suppose the point to begin to move from P; it is required to find the velocity with which it reaches any given point Q. Draw the horizontal lines PN, QM, and the chords AP, AQ. We know from the last article that

$$v^2 = 2g \cdot MN.$$

Also we know from geometry that

$$AP^2 = AB \cdot AN \text{ and } AQ^2 = AB \cdot AM.$$

Therefore $$MN = \frac{AP^2 - AQ^2}{AB} = \frac{AP^2 - AQ^2}{2r},$$

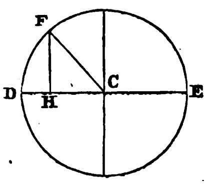
where r denotes the radius of the circle. Hence

$$v^2 = \frac{g}{r}(\text{AP}^2 - \text{AQ}^2).$$

It is plain from this that the velocity at A is $\text{AP} \times \sqrt{\frac{g}{r}}$. In other words, in the same or equal circles the velocity at the lowest point is proportional to the chord of the arc of descent.

127. *Pendulums.*—When a body is suspended, and allowed to swing backwards and forwards, its motion from one extreme position (say, on the right) to the other extreme position (say, on the left) is called an oscillation or a *vibration.* A body when suspended so as to be capable of an oscillatory motion, is called a pendulum, e.g. a small heavy ball fastened to one end of a fine silk thread, the other end of which is fastened to a point of support, is a pendulum. If the thread is so light in comparison with the ball that its weight can be neglected, and the ball so small that it can be treated as a single particle placed at its centre, it is said to be a simple pendulum. All other pendulums are called compound pendulums. In fig. 109 produce PN to meet the circumference in P′. If we suppose the body to be a simple pendulum, whose motion begins at P, the body ascends to P′ and then returns to P. The arc PAP′ is called the arc of vibration. The arc of vibration is said to be *small* when its length does not differ sensibly from that of its chord.

FIG. 110.



128. *Time of an oscillation in a small circular arc.* — Referring to fig. 109, draw the straight like DCE equal to the arc PAP′, and, with the middle point C as centre, describe the circle DFE. Take CH equal to the arc AQ;

draw FH at right angles to DE, and join FC. Now $FH^2 = FC^2 - HC^2 = chd\,.\,AP^2 - chd\,.\,AQ^2$, since the chords are sensibly equal to the arcs, and consequently, if l denotes OP, the velocity of the body at Q equals $FH \times \sqrt{\frac{g}{l}}$. Now suppose a body to move along DE with the same velocity, point by point, as that with which the pendulum moves, it will describe the line DE in a time equal to that of one oscillation. But as the velocity of this body equals $FH \times \sqrt{\frac{g}{l}}$, it will describe DE in the same time as that in which a body moving with a uniform velocity $DC \times \sqrt{\frac{g}{l}}$ describes the semicircle DFE (Art. 116-2). But this time must equal (circumference DFE÷velocity) or $\pi\,.\,DC \div DC\,.\,\sqrt{\frac{g}{l}}$. Hence the time of an oscillation in a small circular arc equals

$$\pi\sqrt{\frac{l}{g}}.$$

If g, the accelerative effect of gravity, is estimated in feet and seconds, l, the length of the pendulum, must be estimated in feet, and then the above formula gives the time of an oscillation in seconds.

Ex. 138.—If a pendulum is 8 ft. long, it will make one oscillation in 1·57 sec., or 38·2 oscillations a minute, if the force of gravity is 32.

Ex. 139.—If a simple pendulum 40·5 in. long were observed to make oscillations a minute at a certain place, what would be the accelerative ffect of gravity at that place?

As the time of one oscillation is $60 \div 59$ in seconds, and the length of the pendulum $40{\cdot}5 \div 12$ in feet, we have to find g from the equation

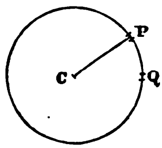
$$\frac{60}{50} = \pi\sqrt{\frac{40{\cdot}5}{12g}},$$

and this gives $g = 32{\cdot}209$.

129. Energy of a rotating body.—Suppose there are

several bodies moving with the same or different velocities, if the energy of each is calculated, and their sum taken, we obtain the energy of the system formed by the bodies. We will apply this simple principle to several cases; and in the first case to a ring revolving uniformly round an axis, passing through its centre at right angles to its plane. Let C be the centre of the ring, on which it turns, and let its radius be denoted by r, its mass by M, and its angular velocity by θ; consider a small portion of the ring such as P, whose mass is m. The energy of this portion is $\frac{1}{2} m r^2 \theta^2$; consider another portion Q, whose mass is m_1, its energy is $\frac{1}{2} m_1 r^2 \theta^2$. Consequently the joint energy of the two is $\frac{1}{2}(m+m_1) r^2 \theta^2$. If we consider a third and a fourth part, we obtain a result differing from that already found only in having the sum of three or four masses instead of two. Hence if we suppose the sum taken for all the parts of the ring, we shall have the total energy equal to $\frac{1}{2} \mathrm{M} r^2 \theta^2$. In a following article we shall see how to extend this method to other bodies; but without going further we will give some applications of the result we have already obtained.

FIG. 111.



Ex. 140.—Suppose the ring to weigh 12 lbs., to have a radius of 2 ft., and to make 100 revolutions a minute; its angular velocity is $10\pi \div 3$, and its energy $\frac{1}{2} \times 12 \times 2^2 \times (10\pi \div 3)^2$, or 2632 (the units being feet and absolute units of force). Assuming the force of gravity to be 32, this energy would be sufficient to raise a pound of matter to a height of 82 ft.; it might be applied to do in any other way 2632 units of work (or about 82 foot-pounds).

Ex. 141.—A fly-wheel weighs 5 tons; its mass may be considered to be distributed along the circumference of a circle 40 ft. in circumference; it turns on an axle 3 ft. in circumference, the coefficient of friction between which and its bearing is 0·05. At a given instant it is moving at the rate of 24 revolutions a minute. Find how many turns it will take before being brought to rest by the friction of the axle,

Each point of the circumference moves at the rate of 24×40 ft. a minute,

and therefore at the rate of 16 ft. a second. The mass is 11,200 lbs.; therefore the energy is $5600 \times (16)^2$. Again, the friction of the axle is $0 \cdot 05 \times 11,200\, g$, and this is overcome through 3 ft. on each turn of the wheel; consequently if the wheel makes x turns in coming to rest, $0 \cdot 05 \times 11,200 \times g \times 3\, x$ units of work must be done against friction. Hence,

$$0 \cdot 05 \times 11,200 \times g \times 3\, x = 5600 \times (16)^2;$$

therefore, $$x = 26\tfrac{2}{3} \text{ turns.}$$

Ex. 142.—In the last case suppose a weight of 1 ton to be fastened to the end of a band which passes round the circumference of the wheel; the weight falls, and so moves the wheel from a state of rest. Find the velocity communicated to the wheel at the end of the first complete turn—all passive resistance being neglected.

Here the work done is $2240\, g \times 40$; if v is the velocity of a point on the circumference of the wheel, v will be also the velocity of the falling mass; hence (Art. 105) the energy of the system is $\frac{1}{2} \times 11,200\, v^2 + \frac{1}{2} \times 2240\, v^2$. As there is no resistance, we must have,

$$\tfrac{1}{2} \times 11,200\, v^2 + \tfrac{1}{2} \times 2240\, v^2 = 2240\, g \times 40.$$

Hence $v = 20 \cdot 66$ ft. per second, i. e. the wheel is moving at the rate of about 31 turns a minute.

Ex. 143.—In Ex. 142 determine the mutual action between the fly-wheel and the descending mass.

In this case, as the system undergoes a uniform acceleration, R must be a constant force. The work done by R in one complete turn of the wheel is $40\,\text{R}$, while the energy of the wheel is $\frac{1}{2} \times 11,200\, v^2$. Hence,

$$40\,\text{R} = 5600\, v^2.$$

By similar reasoning the motion of the falling weight gives the equation

$$40(2240\, g - \text{R}) = 1120 \,.\, v^2.$$

And therefore, $$6\,\text{R} = 2240\, g \times 5,$$

i. e. the mutual action between the descending weight and the wheel equals the force of gravity on 5-6ths of a ton of matter.

130. *Moment of inertia.*—Let several rigidly connected points whose masses are respectively $m, m_1, m_2, \ldots$ rotate about an axis from which their distances are $r, r_1, r_2 \ldots$. Their common angular velocity being θ, their several velocities are $r\,\theta$, $r_1\theta_1$, $r_2\theta_2 \ldots\ldots$ and consequently the energy of the whole system is $\frac{1}{2}(mr^2 + m_1 r_1^2 + m_2 r_2^2 + .)\,\theta^2$. The quantity between the brackets is called the *moment of inertia* of the system

of points with respect to the axis of rotation. As the moment of inertia is formed by adding together the products obtained by multiplying the mass of each point by the square of its distance from the given axis, it is plain that it will equal the product formed by multiplying the total mass of the system by the square of some line, i.e. if k denote the length of the line in question, and M the sum of the masses of the points, it is possible to take k of such a length that

$$\mathrm{M}\,k^2 = m\,r^2 + m_1\,r_1^{\,2} + m_2\,r_2^{\,2} + \ldots\ldots$$

The line whose length is k is called the radius of gyration, with respect to the axis from which the distances $r, r_1, r_2, \ldots\ldots$ are measured. It follows from what was stated at the beginning of the article that the energy of the system equals $\frac{1}{2}\,\mathrm{M}\,k^2\,\theta^2$.

If we supposed a point whose mass is M to revolve about the axis at distance k, and with the same angular velocity as the body, its velocity would be $k\,\theta$, and its energy $\frac{1}{2}\,\mathrm{M}\,k^2\,\theta^2$. We see therefore that the radius of gyration is the distance from the axis at which the mass of a rotating body may be supposed to be concentrated without changing its energy.

131. *Determination of moment of inertia.*—As the number of points considered in the last article may be as large as we please, the definition of the moment of inertia applies to that of any solid body whatever. There are methods by which the moment of inertia of any body of given form can be determined. Thus, the moment of inertia of a straight line or rod with reference to an axis passing through one end at right angles to its length can be shown to equal $\frac{1}{3}\,\mathrm{M}\,l^2$, where M and l denote the mass and length of the rod.[1] When a rod is spoken of without

[1] In fact, conceive the rod to be divided into a large number (n) of equal parts, and consider the case in which n points of equal mass

qualification, it is assumed to be of uniform density, and to have a small uniform cross-section. Of course in the case of the rod the radius of gyration equals $l \div \sqrt{3}$. The two following facts about the moment of inertia deserve notice:—

(*a*) If any body can be conceived to be divided into two or more parts, such that the moment of inertia of each part is known with regard to a given axis, the moment of inertia of the whole with regard to the same axis is the sum of the moments of inertia of the several parts. This is, in fact, plain from the definition, *e.g.* there is a rod 12 ft. long whose mass is 6; required *its* moment of inertia about an axis at right angles to *its* length drawn through a point 3 ft. from one end. Here we have two rods whose masses are $1\frac{1}{2}$ and $4\frac{1}{2}$ respectively, and lengths 3 ft. and 9 ft. Consequently their respective moments of inertia are $\frac{1}{3} \times 1\frac{1}{2} \times (3)^2$ and $\frac{1}{3} \times 4\frac{1}{2} \times (9)^2$, *or* the required moment is 126, and the radius of gyration (being given by the equation $6\,k^2 = 126$) is $\sqrt{21}$, or 4·582 ft. in length.

(*b*) When the moment of inertia of a body is required

$\left(\frac{M}{n}\right)$ are arranged along the line at distances $\frac{l}{n}, 2\frac{l}{n}, 3\frac{l}{n} \ldots n.\frac{l}{n}$ from the end. The moment of inertia of these points will be,

$$\frac{M}{n} \cdot \left(\frac{l}{n}\right)^2 + \frac{M}{n} \cdot \left(\frac{2\,l}{n}\right)^2 + \frac{M}{n} \cdot \left(\frac{3\,l}{n}\right)^2 + \ldots + \frac{M}{n} \cdot \left(\frac{n\,l}{n}\right)^2,$$

or

$$\frac{M\,l^2}{n^3}(1 + 4 + 9 + \ldots + n^2).$$

Now it is well known that $1 + 4 + 9 + \ldots + n^2 = \frac{1}{6}\,n\,(n + 1)\,(2\,n + 1)$, consequently the moment of inertia of the points equals

$$\tfrac{1}{3}\,M\,l^2\left(1 + \frac{1}{n}\right)\ \left(1 + \frac{1}{2\,n}\right).$$

If we suppose n to be indefinitely great, we pass from the case of the points *to that* of the line, and we see that this formula gives for the value of the *moment of inertia*, $\frac{1}{3}\,M\,l^2$, as stated in the text.

with respect to an axis whose distance from the body is several times as long as the greatest dimension of the body, the error is very small which results from treating the body as a point whose mass is collected at its centre of gravity, and consequently its moment of inertia with respect to the axis is $m r^2$, where m is the mass of the body and r the distance of the centre of gravity from the axis, e.g. the radius of a sphere is 1 in. and mass m, the exact value of its moment of inertia about an axis distant 10 in. from its centre can be shown—by a method that need not detain us—to equal $m\,(10^2 + \frac{2}{5} \times 1^2)$, or $100{\cdot}4\,m$. If treated as a point, according to the above remark, the moment of inertia would be taken to equal $100\,m$, which only differs from the true value by the 1–250th part. In the same case the approximate value of the radius of gyration is 10 in., the true value $\sqrt{100{\cdot}4}$, or 10·02 in. If the distance of the ball from the axis had been 20 in., the exact value of the moment of inertia would have been $400{\cdot}4\,m$, instead of $400\,m$, and the radius of gyration 20·01, instead of 20.

Ex. 144.—If a rectangular lamina[1] whose sides are a and b is considered, it will be seen that it can be divided into a number of rods whose lengths are equal to the edge b; consequently it can be easily deduced from section (a) of the last article that its moment of inertia with respect to the edge a is $\frac{1}{3}$ (mass) $\times b^2$, i.e. if m denote the mass of each unit of area, the moment with regard to a equals $\frac{1}{3}\,m\,a\,b^3$. Of course its moment of inertia with respect to the edge b will equal $\frac{1}{3}\,m\,a^3\,b$.

Ex. 145.—The moment of inertia of a rectangular lamina with respect to an axis parallel to the edge a, and passing through its centre of gravity, will be $\frac{1}{12}\,m\,a\,b^3$.

Ex. 146.—The moment of inertia of a thin hollow drum with reference to its axis is (mass) $\times$ (radius)^2.

Ex. 147.—A rectangular door is pushed from rest through an angle of

[1] A lamina is, of course, a slice of any substance. The rectangle in the question is supposed to be heavy, and its thickness uniform, but so small that it can be neglected in comparison with its length and breadth.

90° by a force of 160 absolute units, acting at right angles to its face at a distance of 4 ft. from the axis of rotation. The door is 7 ft. high and 5 ft. wide, and weighs 120 lbs. What angular velocity has the door acquired at the instant the 90° are completed?

The angle in circular measure is $\frac{\pi}{2}$, and consequently the work done by the force is $\frac{\pi}{2} \times 4 \times 160$, or $320\,\pi$; the moment of inertia of the door with respect to the axis—the door being reckoned as a rectangular lamina—is $\frac{1}{3} \times 120 \times 5^2 = 1000$; hence if θ is the angular velocity acquired, we have

$$500\,\theta^2 = 320\,\pi,$$

or

$$\theta = 1{\cdot}42.$$

In other words, if the door were hung in such a way that it could turn round and round on its axis quite freely, it would turn at the rate of about $13\frac{1}{2}$ times a minute after the force had ceased to act.

The pressure on the hinges, so far as it is due to the rotation, would be $120 \times \frac{5}{2} \times \theta^2$, i.e. 603 absolute units, or about 18·8 gravitation units.

Ex. 148.—A trapdoor turns freely on a horizontal spindle; its width at right angles to the spindle is 3 ft., and it weighs 40 lbs. It falls through an angle of 180° from its highest vertical to its lowest vertical position; find its velocity as it passes it lowest position, and the pressure it exerts at that instant on its points of support.

As the centre of gravity falls through 3 ft., gravity does $3 \times 40\,g$ units of work, while the body falls from its highest position to its lowest. As it falls quite freely, this work must equal the energy of the body as it passes through the lowest position; but if θ is the required angular velocity, the energy is $\frac{1}{2} \times \frac{1}{3} \times 40 \times 3^2 \times \theta = 60\,\theta^2$, and therefore

$$60\,\theta^2 = 120\,g;$$

or, if $g = 32$,

$$\theta = 8;$$

in other words, the body is moving at such a rate that it would make 1·273 turns a second, or about $76\frac{1}{2}$ turns a minute, if it continued to revolve uniformly. The centrifugal force as the body passes the lowest point is $40 \times \frac{3}{2} \times \theta^2 = 120\,g$ absolute units; and this added to the weight of the body $(40\,g)$ gives the whole pressure at the instant as $160\,g$ absolute units, or about 160 gravitation units. If we suppose the motion to take place in London, or in any place where the force of gravitation is the same as in London, the pressure as the body passes the lowest point is, of course, 160 gravitation units exactly.

Ex. 149.—A body of any shape turns on a horizontal spindle; it is set in motion by a weight which unwinds from a concentric circle—as a wheel and axle might be set in motion by the descending bucket—determine the angular velocity communicated to the body by the weight, in falling from rest through a height h.

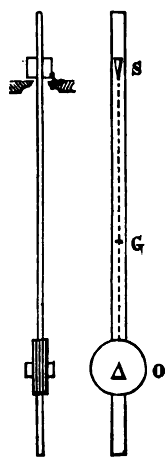
; $M k^2$ denote the moment of inertia of the body, m the mass of the weight, and r the radius of the circle on which the weight acts. If θ ; the angular velocity of the body, $r\theta$ is the velocity of the descending , and consequently we have

$$m g h = \tfrac{1}{2} M k^2 \theta^2 + \tfrac{1}{2} m r^2 \theta^2,$$

ation in which every quantity is known but θ, which it serves to de- ıe.

32. *Compound pendulum.*—Suppose a body of any ; to be furnished with a steel wedge or knife-edge ng through it at one end, as shown so that it could be supported on the of the wedge and made to oscillate— dge is called the centre of suspen- Now if the time of a small oscil- n be observed, it is easy to determine ength of a simple pendulum which es a small oscillation in the same as the compound pendulum (Art. . Suppose the length of the simple ulum thus determined to be l. Take centre of gravity of the pendulum, s G and produce it to O, making S O l to l; the point O is called the centre cillation. These three points, S, G, e always in the same line, and G termediate to S and O. Suppose ody to be furnished with a second wedge or 'knife- ,' capable of being placed in different positions, it lain that the second wedge can by successive trials rought into such a position that its edge passes ıgh O. When this is done it will be found that the will make its small oscillations in equal times, her it be suspended by S or by O. In other words, if he centre of suspension, O is the centre of oscillation, if O is the centre of suspension, S will be the centre of

FIG. 112.



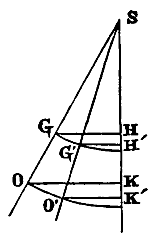
oscillation. A fact which is generally stated thus—the centres of oscillation and suspension are reciprocal.

If k be the radius of gyration of the pendulum with respect to the axis of suspension, it admits of proof that the following relation holds good :—

$$\text{SO} \cdot \text{SG} = k^2.$$

133. The relation stated in the last article can be proved as follows:—Draw the vertical line SK, and let SG be the position of the line SG (in fig. 112) at the beginning of the motion; let SG′ be the position of SG at any subsequent time. Draw the horizontal lines GH, G′H′, OK, O′K′; let θ denote the angular velocity of the body when SG comes into the position SG′; the energy of the body is $\frac{1}{2} \text{M} k^2 \theta^2$, which is wholly due to the action of gravity, viz. $\text{M} g \times \text{HH}'$, and therefore

FIG. 113.



$$k^2 \theta^2 = 2g \times \text{HH}'.$$

If a point were placed at O, and allowed to swing freely round S, and if its angular velocity when at O′ were θ', its velocity would be $\text{SO} \times \theta'$, and therefore (Art. 125)

$$\text{SO}^2 \times \theta'^2 = 2g \times \text{KK}'.$$

Now if the angular velocity of O at O′ is the same as that of the body when at SG′, for all positions of SG′, the time of an oscillation of O is the same as that of the body, i.e. if $\theta = \theta'$, O will be the centre of oscillation. The condition, therefore, of O being the centre of oscillation is that

$$\text{KK}' \times k^2 = \text{HH}' \times \text{SO}^2.$$

But, as the figures are plainly similar, we have

$$\text{KK}' : \text{HH}' :: \text{SO} : \text{SG};$$

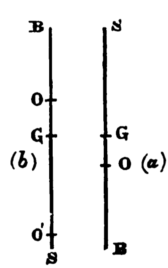
therefore, $$k^2 = \text{SO} \cdot \text{SG}.$$

It of course follows from this that the time of oscillation of the compound pendulum, being the same as that of a simple pendulum, whose length is s o, will equal

$$\pi\sqrt{\frac{\text{s o}}{g}} \text{ or } \frac{\pi k}{\sqrt{g \cdot \text{s G}}}.$$

Ex. 150.—Let s B (fig. 114, *a*) be a rod whose mass and length are denoted by m and l; let it be suspended from s, and allowed to oscillate. We know that k^2 equals $\frac{1}{3}l^2$, and that s G equals $\frac{1}{2}l$; consequently $\frac{1}{2}l \times \text{s o} = \frac{1}{3}l^2$, so that s o $= \frac{2}{3}l$; this gives the position of o, and of course the time of a small oscillation is given by the formula $\pi\sqrt{\frac{2l}{3g}}$, e. g. if we assume g to equal 32, and suppose the rod to be 12 ft. long, the time of an oscillation is 1·5708 sec., i. e. it makes about 38 oscillations a minute.

FIG. 114.



Now suppose the rod to be suspended by o (fig. 114, *b*), and let it be required to determine the corresponding centre of oscillation o′. Since the lengths and masses of o B and o s are $\frac{1}{3}l$, $\frac{2}{3}l$, $\frac{1}{3}m$ and $\frac{2}{3}m$ respectively, the moment of inertia of the rod about o is $\frac{1}{3} \cdot \frac{m}{3} \times \left(\frac{l}{3}\right)^2 + \frac{1}{3} \cdot \frac{2m}{3} \times \left(\frac{2l}{3}\right)^2$, or $\frac{1}{9}ml^2$, and therefore in this case $k^2 = \frac{1}{9}l^2$, also G o $= \frac{1}{6}l$. Now the distance o o′ is determined by the relation o G × o o′ $= k^2$, whence o o′ $= \frac{2}{3}l$, i. e. the point o′ coincides with s, which accords with the rule that the centres of suspension and oscillation are reciprocal.

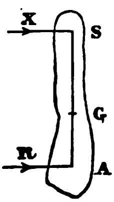
Ex. 151.—A rod 12 ft. long has a small heavy sphere placed on it capable of being shifted up and down; the mass of the sphere is four times that of the rod; the rod is set to oscillate upon an axis passing through one end. Find the number of oscillations it makes per minute when the sphere is placed at the other end of the rod, assuming that the sphere can be treated as a point ($g = 32$).

Here, if we denote by s the centre of suspension, we have the moment of inertia of the rod and ball with respect to s equal to $\frac{1}{3}\text{M} \times 12^2 + 4\,\text{M} \times 12^2$, where M and 4 M are the masses of the rod and ball; consequently $k^2 = \frac{624}{5}$. If G is the centre of gravity of the whole body made up of rod and ball, we have $5\,\text{M} \times \text{s G} = 6\,\text{M} + 12 \times 4\,\text{M}$, or s G $= \frac{54}{5}$ ft. Hence $k^2 \div$ s G equals $11\frac{5}{9}$ ft., and consequently the time of one oscillation is 1·888 sec.

Ex. 152.—In the last case, if we suppose the sphere shifted to the middle of the rod, the time of one oscillation is 1·405 sec.

134. *Motion of a pendulum produced by impact.*—Let the annexed figure represent a body suspended from an axis passing through S at right angles to the plane of the paper, and let us suppose the body to be symmetrical with reference to that plane. Let G be the centre of gravity; draw the line SG, take A any point in SG or its prolongation, and draw RA at right angles to SA. Suppose the body to be struck along the line RA, the body with its axis will then turn on the bearings, and the axis will generally be impelled against the bearings; consequently there will generally be a reaction of the bearings against the body, which will be exerted along a line parallel to AR. Let R and X denote the blow and the consequent reaction, the latter being supposed to act as shown in the figure; then it admits of proof that the following relations hold good:—

FIG. 115.



$$\mathrm{R}+\mathrm{X}=\mathrm{M}\,.\,\mathrm{SG}\,.\,\theta \text{ and } \mathrm{M}k^2\theta=\mathrm{R}\,.\,\mathrm{SA},$$

where M denotes the mass of the body, $\mathrm{M}k^2$ its moment of inertia with respect to the axis, and θ the angular velocity generated by the blow, which is determined by the latter equation. When θ is known, the former equation gives X, the reaction of the bearings against the body. In any example, if X comes out negative, the reaction is in a direction opposite to that shown in the figure.

Ex. 153.—A rod 12 ft. long whose mass is 18 lbs. is suspended by one end and is struck at the other (in a direction at right angles to its length) a blow measured by a momentum 90. Find the angular velocity with which the rod begins to turn round the point of support, and the impulse of the rod against the point of support.

A blow 'measured by a momentum 90' may be conceived thus:—A very large force acts for an exceedingly short time, such that if we suppose it to act for an equal time on a mass of 1 lb. (perfectly free to move), it would communicate to the mass a velocity of 90 ft. a second.

In this case the moment of inertia equals $\frac{1}{3}\times 18\times 12^2$.

Therefore, $\frac{1}{3}\times 18\times 12^2\theta=90\times 12$;

so that $\theta = \frac{5}{4}$, i.e. if the body continued to revolve uniformly with the velocity imparted by the blow, it would make one revolution in $2\pi \div \frac{5}{4}$, or 5·03 sec., or it would revolve rather less than 12 times a minute. Next, as SG equals 6 ft.

$$X + 90 = 18 \times 6 \times \tfrac{5}{4} = 135,$$

i. e. $X = 45$, which is the reaction against the body at the point of support; it is in the same direction as the blow and of half the amount, so that the rod, in consequence of the blow at one end, is instantaneously struck at the other by a parallel force in the same direction, and of half the amount. The point of support is struck by the rod in the opposite direction with a force equal to X or 45.

Ex. 154.—Suppose the trapdoor in Ex. 148 to strike against an obstacle distant 3 ft. from the spindle when it reaches its lowest position. Determine the magnitude of the blow and of the impulse on the axis, assuming that there is no rebound.

The reaction of the obstacle must be such as to generate in the opposite direction an angular velocity equal to that with which the body reaches its lowest position; consequently, using the data of Ex. 148, we have

$$\tfrac{1}{3} \times 40 \times 3^2 \times 8 = R \times 3,$$

or $R = 320$ (i. e. the blow is such as would communicate to a pound of matter a velocity of 320 ft. a second). Also,

$$320 + X = 40 \times \tfrac{3}{2} \times 8 = 480,$$

i. e. $X = 160$; or the bearing reacts on the spindle with a force equal to half that with which the body strikes against the obstacle. If it were asked, could a point be found such that if this body strike against it, there will be no impulse on the bearing? we may proceed thus:—Suppose x to be the distance from the spindle of such a point then, since $X = 0$, we have

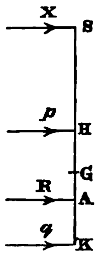
$$R = 40 \times \tfrac{3}{2} \times 8 \text{ and } \tfrac{1}{3} \times 40 \times 3^2 \times 8 = R\,x,$$

i. e. $R = 480$ and $x = 2$; in other words, if the obstacle is 2 ft. from the spindle, it takes the whole force of the blow, and there is no impulsive pressure of the spindle against its bearings. The obstacle must, of course, be in the line drawn at right angles to the spindle through the centre of gravity.

135. To illustrate the reasoning by which the results in Art. 134 are obtained, we will consider the following case:—Let p and q be two forces acting at H and K, X and R two other forces acting at S and A; all at right angles to SK; now suppose that the resultant of X and R is identical with that of p and q; we must then have the

following relations: the resultant must equal X + R, and likewise $p+q$ (Art. 46);

FIG. 116.



therefore, $$X+R=p+q.$$

Also the sum of the moments of p and q with regard to any point must equal the moment of the resultant with regard to the same point, and the like is true of X and R (Art. 48); hence if we take moments with respect to S we must have

$$SA \,.\, R = SH \,.\, p + SK \,.\, q.$$

We may suppose the forces measured by the momenta they would produce in an exceedingly short time; this would introduce no change into the above relations. Now suppose that SK is a rigid rod without inertia, capable of turning round the point S, and that H and K are two points whose masses are m and n respectively; and suppose that R and X are such as, acting for a very short time, will cause the body to move round S with an angular velocity θ. Let us denote SH and SK by a and b; then the velocities of H and K are $a\,\theta$ and $b\,\theta$, and their momenta $m\,a\,\theta$ and $n\,b\,\theta$; as these are the actual effects of X and R at H and K, they must be equal to p and q. Consequently the above equations become

$$X+R=m\,a\,\theta+n\,b\,\theta \text{ and } R\,.\,SA=m\,a^2\,\theta+n\,b^2\,\theta.$$

If R is given, we find θ and X from these equations; X is the force which must be exerted at S simultaneously with R, if the body is constrained to turn round S; it is therefore the reaction of the bearing against the body. That these equations are only a particular case of those given in the last article can be shown thus:—Let G be the centre of gravity of H and K, and M their joint mass. Then we have (Art. 15)

$$M\,.\,SG=m\,a+n\,b.$$

Also if $\text{M}\,k^2$ is the moment of inertia of the two points with reference to S, we must have (Art. 130)

$$\text{M}\,k^2 = m\,a^2 + n\,b^2,$$

and consequently the above equations can be written

$$\text{X} + \text{R} = \text{M} \,.\, \text{SG} \,.\, \theta \text{ and } \text{R} \,.\, \text{SA} = \text{M}\,k^2\,\theta,$$

i.e. they are a particular case of the equations given in Art. 134. Similar reasoning can be applied to three, four, or any number of rigidly connected points. When, however, this is done, it will appear that the impact against the point of support will be represented by a single force X, whenever the body is symmetrical with reference to a plane passing through G at right angles to the axis (the plane of the paper in fig. 115) and not generally in other cases. The body is said to be symmetrical with regard to a plane when it would be divided into two parts exactly alike if it were cut by the plane; in other words, when the body is such, that if any point whatever of it is considered, there will always be a second point such that the line joining them will be at right angles to the plane, and will be bisected by it, e.g. a sphere is symmetrical with regard to any plane containing a diameter; a cylinder or cone with regard to any plane containing its axis; a cube with regard to a plane bisecting any four parallel edges; a hammer with regard to a plane containing the axes of head and handle.

136. *Centre of percussion.*—In Art. 134 let us assume that the point A has been so chosen that X is equal to zero, we then have two equations:—

$$\text{R} = \text{M} \,.\, \text{SG} \,.\, \theta \text{ and } \text{M}\,k^2\,\theta = \text{R} \,.\, \text{SA},$$

which give $$k^2 = \text{SG} \,.\, \text{SA};$$

and this equation tells us where A must be taken in the body that X may equal zero. In all cases in which the

whole reaction of the bearing against the body would be X, there will be no impulse against the bearing when the point A is thus determined. This will be the case whenever the body is symmetrical with regard to the plane of the paper. The point thus determined is called the *centre of percussion.* It is evident from Art. 132 that the centre of percussion (when there is one) with regard to S coincides with the centre of oscillation.

Ex. 155.—If a rod 10 ft. long whose mass is 20 lbs. has a point weighing 20 lbs. firmly fixed to one end and hangs by the other, its moment of inertia is $\frac{1}{3} \times 20 \times 10^2 + 20 \times 10^2$; also if x is the distance of its centre of gravity from the point of suspension, $40\,x = 20 \times 5 + 20 \times 10$. The distance of the centre of percussion from the same point is therefore $80 \div 9$ ft. Suppose now, that a horizontal blow is struck through the centre of percussion such as to make the body just rise to a horizontal position, and let it be asked what is the magnitude of the blow? We can reason thus:—Let θ be the angular velocity communicated by the blow; then, as the body leaves its first position, its energy ($\frac{1}{2}$ M $k^2\theta^2$) is $\frac{1}{3} \times 4000\ \theta^2$; and this is just sufficient to raise the body to a horizontal position, i. e. to do $20\,g \times 5 + 20\,g \times 10$ units of work, so that θ^2 equals $9g \div 40$, or θ equals 2·683. Hence if R denotes the momentum of the blow as

$$\text{M}\,k^2\theta = \text{R} \,.\, \text{SA},$$

we have

$$\text{R} \times \frac{80}{9} = \frac{4000}{3} \times 2{\cdot}683,$$

or

$$\text{R} = 402{\cdot}5,$$

i. e. the blow must be nearly as great as one which gives to a weight 10 lbs. a velocity of 40 ft. a second.

137. *Newton's laws of motion, and proof of the parallelogram of forces.* —Newton states and illustrates the laws of motion as follows:—[1]

1. Every body continues in its state of rest or of uniform motion in a straight line, except so far as it is compelled by impressed forces to change its state.

Projectiles continue in their state of motion, except so far as they are retarded by the resistance of the air, and urged downward by the force of gravity. The parts of a hoop by their cohesion continually draw one another back from their rectilinear motions; the hoop, however, does not cease to turn, except so far as it is retarded by the air. Moreover the greater bodies of planets and comets, which move in spaces offering less resistance than the air, preserve for a longer time their motions of translation and rotation.

[1] *Principia*, p. 13 (3rd edition).

2. Change of momentum is proportional to the impressed moving force, and takes place along the straight line in which that force is impressed.

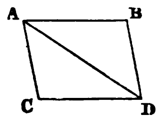
If a force produce any momentum whatever, twice the force will produce twice the momentum, thrice the force will produce thrice the momentum, whether the forces are impressed at the same time and instantaneously, or gradually and successively. This momentum is always communicated in the same direction as the force producing it; so that, when the body was previously moving, it is added to the body's momentum if the directions are the same, subtracted from it if the directions are opposite; it is added obliquely if the directions are inclined; and the momenta are compounded in accordance with their several directions.

3. Reaction is always contrary and equal to action; or the mutual actions of two bodies on each other are always equal and exerted in contrary directions.

If any body presses or draws another, it is just as much pressed or drawn by the second body. If anyone presses a stone with his finger, the finger is also pressed by the stone. If a horse is drawing a stone tied to a rope, the horse is (so to speak) equally drawn back towards the stone; for the rope being stretched both ways will, in the same endeavour to slacken itself, urge the horse towards the stone, and the stone towards the horse; and it will impede the progress of the one as much as it advances the progress of the other. If a body strike on another body, and by its force change the momentum of the latter body in any way whatever, its own momentum will in turn undergo an equal change in a contrary direction from the force of the latter body, in consequence of the equality of the mutual pressure. By these actions equal changes are produced, not in the velocities, but in the momenta, i. e. in bodies otherwise free to move. The changes of velocities which also take place in opposite directions are reciprocally proportional to the masses; because the momenta are changed equally.

Newton gives as a corollary to the laws of motion the following proof of the parallelogram of forces:--Suppose that a force M, impressed separately on a body when at the point A, would cause it to move in a given time from A to B with a uniform velocity; and suppose that a force N, impressed separately on the same body when at the point A, would cause it to move in the same time from A to C with a uniform velocity. Let the parallelogram ABDC be completed. Then, if the forces are supposed to be impressed simultaneously, the body will move in the same time with a uniform velocity from A to D. For since the force N acts along AC parallel to BD, this force by the second law will not change the velocity of approach to BD, generated by the other force. Therefore the body approaches the line BD in the same time, whether the force N be impressed or not; and at the end of the given time it will be found somewhere in BD. In the same way, it can be proved that at the end of the same time it will be found somewhere in the line CD; and

FIG. 117.



therefore it must be found at their point of intersection D. But by the first law, it will proceed by a rectilinear motion from A to D. [Now bearing in mind that forces impressed instantaneously on equal masses are proportional to the velocities they generate, it follows that the forces M, N, and their resultant are proportional to the lines A B, A C, and A D.]

§ 138. *Newton's experiments on collision.*[1]—Let a pendulum, hung on a point C, swing from an extreme position R, and let A be the lowest point of the arc of vibration. The velocity of A is proportional to the chord A R (Art. 126), and the momentum to the product of the weight of A, and the length of the chord. This would be true in vacuo; the resistance of the air, however, has a sensible effect, and may be taken account of thus. Let A, after leaving R, return at the end of the first double oscillation to V. Take S T a quarter of R V, and put it in the middle of R V, i. e. make S R and T V each $\frac{3}{8}$ of R V. Now, if the body were allowed to fall from S, it would reach A with very nearly the same velocity as if it had fallen in vacuo from T, and accordingly its momentum at A is proportional to the product of the weight and the chord A T. In a like manner, if a body leaving A with a certain velocity is found to reach a point *s*, it would be possible by trial to find the point *t*, to which the body would have risen but for the resistance of the air; T and *t* may be called the corrected positions of S and *s* respectively. Take C and D, two points in the same horizontal line, and let two balls be hung from C and D in such a manner that their centres A and B are in the same horizontal line, and on the wall, in front of which they are supposed to hang, describe semi-circles E A F and G B H, with C and D for centres. Let the body A be allowed to fall from S, and to strike B; after the impact let A rise to *s* and B to *k*. Take T, *t* and *l* the corrected positions of S, *s* and *k*. Then the momentum of A before impact and the momenta of A and B after impact are measured by weight of A × *chd*. A T, weight of A × *chd*. A *t*, and weight of B × *chd*. B *l*. If the principle laid down in Art. 112 be true, the former product ought under all circumstances to equal the sum of the two latter. Or, if the body A had rebounded, so that *s* were found in the arc A R, the former product would equal the difference between the latter products. Now this was verified by

FIG. 118.

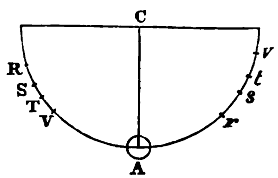
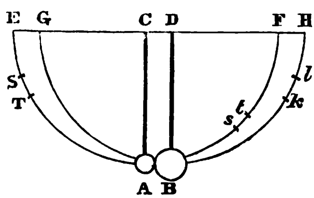


FIG. 119.



[1] *Principia*, pp. 22-3-4 (3rd edition).

Newton's experiments whether the bodies were equal or unequal, hard or soft, whether only one were moving, or both were moving in the same or in opposite directions. His experiments were made with pendulums 10 ft. long, which were allowed to swing 8, 12, or 16 ft. Under these circumstances, he found that with care he could determine the chords without an error of as much as 3 in. He suggests that these errors were due: (1) To the difficulty of letting the body fall, so that the impact should take place exactly at the lowest points of the arcs; (2) to the difficulty of marking exactly two points s and k to which the bodies came after impact; (3) to irregularity of texture in the bodies themselves.

He also determined the ratio of the relative velocity with which the bodies approached each other before impact to the relative velocity with which they separated from each other after impact, and found it to be constant for given substances; e.g. when the balls were of wool firmly compressed this ratio was 5 to 9; when they were of iron the ratio was found to be very nearly the same; when of cork, a little less, and when of glass it was found to be 15 to 16.

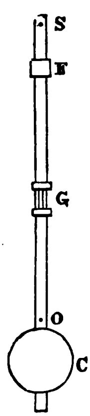
139. *Determination of the accelerative effect of gravity.*—An experimental verification has already been given of the fact that at a given place the force of gravity on bodies is proportional to their masses (Art. 101). The significance of this fact may be exemplified thus:—Suppose two bodies to be taken one of which has double the weight of the other as ascertained by a balance, then if these two bodies were allowed to fall freely from rest at a given place for a given time, the momentum imparted by gravity to the one would be double that imparted by gravity to the other; and the like would be true in other cases. And accordingly at any given place gravity will generate in all falling bodies equal velocities in equal times, quite irrespectively of their being of one kind of matter or another. In order to verify this point with the greater accuracy, Newton made a series of experiments which may be briefly described thus:—He took two equal round wooden boxes, and filled the one with wood, and in the other he placed an equal weight of gold as nearly as possible in the centre of oscillation. The boxes hanging from threads 11 ft. long were pendulums exactly equal in regard to weight, form, and resistance of air. When placed near each other, they continued to go backward and forward for a very long time in equal oscillations; he repeated the experiments with silver, lead, glass, sand, common salt, water, corn, and always with the same result, though an error less than the one-thousandth part could have been detected.[1]

The amount of the accelerative effect of gravity (g) at a given place can also be determined by the pendulum. It can easily be found with some

[1] *Principia*, p. 400 (3rd edit.). The form in which Newton states the principle verified by these experiments is that the weights of bodies at equal distances from the centre of a planet are proportional to the quantities of matter in them.

degree of exactness thus :—Take a ball of lead or brass and fasten it to one end of a very fine silk thread, suspend it by means of a loop at the other end passed over a needle stuck into a wall; set it swinging, and count the number of small oscillations made in two or three minutes; measure the distance from the centre of the ball to the point of support; as the pendulum very nearly realises the condition of a simple pendulum, the value of g can be found from the formula $g\,t^2 = \pi^2\,l$ (Art. 128), e. g. such a pendulum 6 ft. $4\frac{3}{4}$ in. long was found to make a hundred oscillations in 2 min. 20 sec.; this gave $g = 32{\cdot}206$ in feet and seconds, the correct value of g in London being 32·1912. The student should try this with several different lengths; he will easily satisfy himself that the value of g is certainly between 31 and 33, and does not differ much from 32. He will also find that the difficulty in making an exact determination chiefly lies in measuring the exact length of the pendulum, and ascertaining the exact time of a vibration.

Fig. 120.



One very exact method of determining the value of g may be briefly described thus : = s o is a brass plate $1\frac{1}{2}$ in. broad, $\frac{1}{8}$ in. thick; c, a large flat bob; F, a weight capable of adjustment, and G a smaller weight capable of exact adjustment by means of a screw; at s is a knife-edge; the position of the centre of oscillation of the pendulum with regard to s can be calculated, and thus a knife-edge can be placed at o, so that o is very nearly the centre of oscillation with regard to s. By causing the pendulum to vibrate first on s and then on o, any error can be detected, and then by adjusting F and afterwards G, the pendulum can be made to oscillate in exactly the same time, whether suspended from s or o, and now s o is the exact length of the simple pendulum, oscillating in the same time as the compound pendulum. Moreover, as s and o are two rigidly connected points, the distance between them can be measured with extreme exactness.

In order to ascertain with exactness the time of an oscillation, the means adopted were as follows :—Place the pendulum, which we will call A, in front of a clock pendulum, which we will call B, and suppose the pendulums to oscillate in nearly the same time, but A rather faster than B; at any instant let them pass the lowest points of their arc of vibration together; the next time A passes a little before B and still more the third time, until, after a considerable number of oscillations, they both pass the lowest point together, moving in the same direction. When this happens A has made exactly one oscillation more than B; the number of oscillations made by B can be read on the dial-plate, and thus any error in counting is avoided. Suppose that B makes 600 oscillations between the two successive coincidences; it follows that in the same time A makes 601 oscillations. The *clock being* kept right by means of astronomical observations, the time in

which the 601 oscillations are made is known exactly. And thus we have a very accurate determination of the time of one oscillation. Corrections have now to be applied, e. g. for length of arc of vibration, buoyancy of atmosphere, temperature, &c.; and thus, in the above formula, l and t are known with the utmost exactness, and g can be found. The following Table gives a number of first-rate determinations [1] of the length of the simple pendulum beating seconds at different stations, and the inferred values of the accelerative effect of gravity at those stations.

TABLE OF ACCELERATIVE EFFECTS OF GRAVITY.

Observer	Station	Latitude	Length of pendulum	G. in ft. and seconds
		° ′	in.	
Sabine	Spitzbergen . .	79 50 N	39·21469	32·2528
Sabine	Hammerfest . .	70 40	39·19475	32·2364
Svanberg	Stockholm . .	59 21	39·16541	32·2122
Bessel	Königsberg . .	54 42	39·15072	32·2002
Sabine	Greenwich . .	51 29	39·13983	32·1912
Borda and Biot . .	Paris	48 50	39·12851	32·1819
Biot	Barcelona. . .	41 23	39·10432	32·1620
Sabine	Sierra Leone . .	8 30	39·01997	32·0926
Sabine	St. Thomas . .	0 25	39·02074	32·0933
Sabine and Duperrey	Ascension . . .	7 55	39·02363	32·0957
Freycinet and Fallows	Cape of Good Hope . .	33 55	39·07800	32·1404

It will be observed on inspecting the above Table that, in proceeding from the north to the equator, the force of gravity continually diminishes, and that it increases in proceeding from the equator southward. This is chiefly due to two causes: (1) As the earth differs sensibly in form from a sphere, it exercises a sensibly different attraction at different points of its surface; (2) the diurnal revolution of the earth causes bodies on the earth's surface to describe circles, and consequently some part of the force of the earth's attraction is employed in keeping bodies in these circles; the remainder is the sensible force of gravity.

It may be well to remind the student that when the force of a gravity at a given station is denoted by g, we may interpret the statement in two ways: (1) That the sensible attraction of the earth on a pound of matter at that station is g absolute units; (2) That the accelerative effect of gravity at that station is to communicate in each second to a body falling freely an additional velocity of g ft. a second. Thus, at Königsberg the force of gravity is 32·2002, which shows that: (1) At that place the sensible attraction of the earth on one pound of matter is a force of 32·2002 absolute units; (2) the accelerative effect of gravity at that place is to communicate in each second to a body falling freely an additional velocity of 32·2002 ft. per second.

[1] Taken from the Table in Airy's *Figure of the Earth*, p. 229.

QUESTIONS.

1. State the rule for the composition of velocities, and give an instance showing the truth of the rule.

2. A ship sails due north at the rate of $7\frac{1}{2}$ miles an hour; a man walks across its deck—in a direction due west—at the rate of 330 ft. a minute; find the velocity and direction of his motion.
Ans. 12·3 ft. a second; 26° 34′ W. of N.

3. A body moves from a certain point O along a line O A, at the rate of 2 ft. a second; at the end of the 2nd second a velocity of 3 ft. a second is impressed on it, in a direction at right angles to O A; find the distance of the body from O at the end of two more seconds? *Ans.* 10 ft.

4. Explain how a body can be made to describe the sides of a regular polygon with a constant velocity by having a certain velocity impressed on it at each angular point.

5. When a body describes the sides of a regular octagon with a constant velocity of 2 ft. a second, what must be the magnitude of the velocity impressed at each angular point? *Ans.* 1·43 ft. a second.

6. A body moves with a uniform velocity in a circle; find the components of the velocity at any point parallel, and at right angles to a given diameter.

7. Find the points at which the velocities parallel to the given diameter are equal to $\frac{2}{3}$, $\frac{1}{2}$, and $\frac{1}{3}$ of the velocity of the body.

8. When a body describes a circle with a uniform velocity, what must be the force (or resultant of the forces) acting on the body, and in what direction and along what line does it act?

9. A body weighing 15 lbs. is tied to the end of a thread 3 ft. long, and is whirled round in a circle 80 times a minute with a uniform velocity; what force must act on the body? *Ans.* 3158 abs. un.

10. What is the quantity of matter in a body which, tied to the end of a thread, could be just supported against gravity in London, by a force equal to that determined in the last question? *Ans.* 98·11 lbs.

11. Give the reasoning by which it is shown that, when a body moves with a uniform velocity in a circle, (1) the force must act along the radius towards the centre, (2) and equals $M\,V^2 \div r$.

12. What is meant by the centrifugal force of a body moving in a circle? On what does it act? Give an example to show the distinction between the centrifugal force of the body and the force which makes the body move in the circle.

13. Imagine a wheel with smooth spoke placed with its plane horizontal and axle vertical; a smooth ring weighing 4 lbs. is placed on one of the

spokes; if the wheel is made to revolve, it will be found that the ring will slide along the spoke till it reaches the rim; why will it do this? If the rim is 3 ft. from the centre, and the wheel makes 4 revolutions a second, show that the mutual actions of the ring and the rim are forces of 7580 absolute units (235·5 gravitation units). Of these two actions, which is the centrifugal force?

14. What is the centrifugal force of a pound of matter at the equator of the earth (see Q. 5, Chap. 5)? *Ans.* 0·1113 abs. un.

15. Given that the accelerative effect of gravity at the earth's equator is 32·0933, what would it be if the earth had no motion of rotation?

Ans. 32·2046.

16. A body is fastened to the outside of the rim of the wheel in Q. 13, in such a way that a force of 12 gravitation units (say 384 absolute units) is required to detach it; how many turns must the wheels make a minute that the body may be just detached by its centrifugal force? *Ans.* 54.

17. A body describes a circle 10 ft. in radius under the action of a constant force of 100 absolute units, tending to the centre; what is the energy of the body? and if its mass is 0·2 pounds, what is its velocity?

Ans. (1) 500 u. w. (2) 70·7 ft. a second.

18. A body moves in a circle whose radius is 10 ft., its energy undergoes a change of from 320 to 960 u. w., show that its centrifugal force changes from 64 to 192 absolute units.

19. A wheel with smooth spokes 8 ft. long revolves in a vertical plane, a smooth ring weighing 7 lbs. is placed on one of the spokes, the end of the spoke revolves with a velocity of 16 ft. a second; what pressure does the ring exert against the rim of the wheel (1) as it passes the highest, (2) as it passes the lowest point ($g = 32$)? *Ans.* (1) 0, (2) 14 lbs. (approx.)

20. What is meant by the angular velocity of a rotating body? If a body makes three complete revolutions in a second, what is its angular velocity? If the rotating body moves with a constant velocity such that the radius of any point describes an angle of 50° a second, what is its angular velocity? *Ans.* (1) 6π, (2) $\frac{5\pi}{18}$.

21. If m is the mass of a point moving with an angular velocity θ, at a perpendicular distance, r, from the axis of rotation, what are its velocity, momentum, energy, and centrifugal force?

22. A number of bodies (which can be treated as points) weighing 5 lbs. a piece are placed on a rod (like beads), the rod (supposed to continue in one plane) turns round one end at the rate of 90 turns a minute; determine

the velocity, momentum, energy, and centrifugal force of a point distant 3 ft. from that end.

Ans. (1) 28·3 ft. a second; (2) 141·4 lbs. and ft. a second; (3) 1998·6 abs. un. and ft.; (4) 1332·4 abs. un.

23. When a thin plate of matter revolves round an axis at right angles to its plane, how does it appear that its centrifugal force is the same as if its whole mass were concentrated at its centre of gravity? Mention some other cases in which the same is true.

24. Two balls (A and B), weighing 200 lbs. and 100 lbs., have their centres joined by a line A B 10 ft. long; they turn once a second round an axis passing at right angles through the middle point of A B; find the centrifugal force. *Ans.* 19,739·2 abs. un.

25. Draw two lines A B and A C inclined at any angle, and suppose that A B represents a rod, which is caused to rotate round A C as an axis; why does the rotation tend to make A B come into a position at right angles to A C?

26. In the last question, suppose A B to be 10 ft. long, and to weigh 20 lbs.; also suppose it to be divided at the middle point, and the two parts connected by a very short thread; when A B is at right angles to A C, and the rod turns 120 times a minute, what is the tension of the thread?

Ans. 11,843·5 abs. un.

27. If the gate in Ex. 79 were to swing with an angular velocity 3, what effect would this produce on the pressures on the hinge and turning point, the centre of gravity being supposed to be on a horizontal line midway between the hinge and turning point ($g=32$)?

Ans. Pressure on turning point, $61\frac{1}{4}$ lbs.; horizontal pull on hinge, $218\frac{3}{4}$ lbs.

28. In the last question, show that the effect of increasing the angular velocity to 4 would be to reduce the pressure on the turning point to zero, and increase the horizontal pull on the hinge to 280 lbs.

29. State what is meant by a permanent axis of rotation, and the distinction that exists between two classes of permanent axes of rotation.

30. When a body slides down a smooth curve under the action of gravity, determine the relation between the velocities at two assigned points. Would the same relation hold good if the curve were rough? Would it in the case of a body tied to a thread and swinging in a circle?

31. A body moves in a smooth vertical circle, whose radius is 12 ft.; at the highest point it has a velocity of 24 ft. a second; what will be its velocity at the lowest point ($g=32$)? *Ans.* 46 ft. a second.

32. In the last question, what is the centrifugal force of a body weighing 1 lb. at the highest and at the lowest point?

Ans. (1) 48; (2) 176 abs. un.

33. If a body is tied to a thread and whirled round in a vertical circle under such circumstances that its velocity at the highest point of the circle is $\sqrt{gr}$; show that the tension of the thread equals zero as the body passes the highest point, and $5\,\text{M}g$ as it passes the lowest point.

34. Draw two vertical circles, the radius of the one being nine times that of the other; let A and B be the lowest points of the circles; take a point P in the former and Q in the latter, such that the chord AP is twice the chord BQ; if a body slides down the curve from P to A, show that it will only have $\frac{2}{3}$rds of the velocity which a body has at B after sliding from Q to B (the curves are supposed to be smooth).

35. A body is tied to the end of a string, and swings backward and forward through an arc of 120°; show that its centrifugal force as it passes the lowest point equals its weight ($\text{M}g$).

36. In the last question, what will be the tension of the thread at the lowest point?

37. What is a pendulum? What is the distinction between a simple and a compound pendulum? What is the arc of vibration of a simple pendulum? What is meant when the arc of vibration is said to be small?

38. Give the reasoning by which it can be shown that the time of a small oscillation of a simple pendulum equals $\pi\sqrt{l \div g}$.

39. A pendulum makes 60 oscillations a minute at a place where g equals 32; if it were possible to move it to a place where it would make 50 oscillations a minute, what would be the accelerative effect of gravity at that place? *Ans.* $22\frac{2}{9}$.

40. The number of oscillations made in the same time by two pendulums (A and B) are as 10 to 11; show that the ratio of A's length to B's length is 121 : 100.

41. When a ring of matter turns round an axis passing through its centre in a direction at right angles to its plane, what is its energy?

42. In Ex. 140, if the ring were exposed to a constant resistance of 50 absolute units acting tangentially on its circumference, show that it would make 4·19 turns before being brought to rest, and that at the completion of the second turn its angular velocity would be 7·57.

43. In Ex. 141, show that 1·22 of a horse-power must be used to keep the fly-wheel in a state of uniform rotation (assuming, in strictness, that the force of gravity is the same as at London).

44. In Ex. 141, show that the angular velocity at the end of the first ten turns is $2\pi \div \sqrt{10}$.

45. A ring, as in Ex. 140, weighs 50 lbs., and has a radius of 5 ft.; a weight of 50 lbs. is firmly fastened to the highest point of the ring, and is then allowed to fall, of course making the ring turn on its axis; with what velocity does the body pass the lowest point? If at that point the

weight were detached, how many turns would the ring make in the next minute ($g = 32$, all passive resistances neglected)?

Ans. (1) 17·9 ft. a second; (2) 34·2.

46. What is meant by the moment of inertia of a body with respect to an assigned axis? What by the radius of gyration? How does it appear that the energy of a rotating body is $\frac{1}{2} M k^2 \theta^2$?

47. A rotating body has a mass of 50 lbs.. and an energy of 10,000 units of work when it makes one revolution a second. If the matter in this body were distributed along the circumference of a circle whose centre is in the axis of revolution and plane at right angles to the axis, what must be the radius if, with the same angular velocity, the ring has the same energy as the body? What is the radius of the ring called? *Ans.* 3·18 ft.

48. Give an expression for the moment of inertia of a rod, with reference to an axis passing at right angles to its length through one end. Mention two facts which are often of use in determining the moment of inertia of bodies, and give an example of the use of each.

49. In Ex. 149, suppose the body to have a moment of inertia with respect to the axis of rotation of 8000 (lbs. and ft.), and the falling body to have a mass of 100 lbs. and to act on a circle whose radius is 1 ft.; if it fall from rest through 49 ft., show that it communicates an angular velocity $6\frac{2}{9}$ to the rotating body.

50. In the last example show that the mutual action between the rotating and falling body is a force of 3160·5 absolute units, and that the latter falls through the 49 ft. in $15\frac{3}{4}$ sec. ($g = 32$).

51. Define the terms centre of suspension and centre of oscillation. What is meant by the rule that in a compound pendulum these centres are reciprocal? What relation exists between the distances of the centres of gravity and of oscillation from the centre of suspension, and the radius of gyration with regard to the axis of suspension? Give the proof that this relation exists.

52. If a body makes 20 small oscillations a minute about a horizontal axis, show that the centre of oscillation is 29·18 ft. below the axis ($g = 32$).

53. If, in the last question, the centre of gravity is 20 ft. below the axis of suspension, what is the length of the radius of gyration with respect to that axis? And if the mass of the body is 80 lbs., what is its moment of inertia? *Ans.* (1) 24·16 ft.; (2) 46,689.

54. If k denotes the radius of gyration of a compound pendulum with respect to an axis of suspension (s) and k_1 that of a body with respect to a parallel axis through the centre of oscillation (o), show that $k^2 + k_1^2 = (\text{s o})^2$.

55. When a hanging body is struck in a horizontal direction, what are the relations existing between the momentum of the blow, the angular

velocity communicated, the reaction of the bearings against the body, and the dimensions and mass of the body?

56. A rod whose mass is 20 lbs. and length 10 ft., hangs vertically by one end; it is struck at the other end in a horizontal direction a blow which causes it to begin to move with an angular velocity 3; what was the momentum of the blow? What was the impulse of the body against the bearings? *Ans.* (1) 200, (2) 100 (opposite to direction of blow).

57. In the last question, if a body perfectly free to move, whose mass is 3 lbs., were acted on by the same force and for the same time as the lower end of the rod, with what velocity would it begin to move?

Ans. $66\frac{2}{3}$ ft. a sec.

58. The relations defined in Arts. 134–5–6 are stated to be true when the body in question is symmetrical with reference to a certain plane; what is meant by symmetry in this sense?

59. Verify the rules of Art. 134 by reasoning from first principles on the case of two heavy points connected with the point of suspension by a rigid weightless rod. Extend the reasoning to three or more points similarly joined.

60. Define the centre of percussion. How is its position determined when there is one? Mention some cases in which a body will have a centre of percussion.

61. In Ex. 155, if the blow had been struck through the heavy point, and had been just sufficient to bring the body into a horizontal position, what must have been the momentum of the blow, and what the reaction of the bearing against the body ($g=32$)?

Ans. (1) 716; (2) 89·5, both impulses in the same direction.

62. If a rod with a heavy point on it swings freely about an axis passing through one end, and if the point is made to strike against an obstacle, so as to cause no jar on the axis; show that the point must be placed at a distance of two-thirds of the length of the rod from the axis, whatever be the masses of the rod and the point.

63. State Newton's laws of motion, and give his illustrations of them. Give Newton's proof of the parallelogram of forces.

64. When a body fastened by a thread is allowed to oscillate, how can its momentum at the instant of passing the lowest point be determined, and how corrected for resistance of the air?

65. Describe briefly Newton's experiments on the impact of balls, and state the facts which his experiments verified.

66. Describe briefly Newton's experimental verification of the fact that at a given place the force of gravity on bodies is proportional to their masses.

67. Describe a method by which the accelerative effect of gravity may be found with some degree of exactness. On trying this method, what two points are there, the exact determination of which obviously present difficulties?

68. Describe a method by which an exact determination of the accelerative effect of gravity can be made, and point out how by this method the two difficulties adverted to in the last question are overcome.

69. What two principal causes produce the variations in the numerical value of g observed as we pass from one point of the earth's surface to another?

70. When the numerical value of g at any place is determined, what meanings may be assigned to the number?

71. If weights were ascertained by a spring-balance, what quantity of matter would weigh as much at St. Thomas, as 1000 lbs. of matter weigh at Königsberg? *Ans.* 1003·33.

72. If bodies fell freely in vacuo for a quarter of a minute, how much further would they fall at Spitzbergen than at Sierra Leone?

Ans. 18. ft.

73. Two pendulums oscillate in exactly 2 sec., one at Spitzbergen, the other at Ascension; how much longer is the former than the latter?

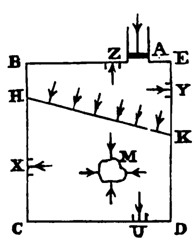
Ans. 0·76424 in.

CHAPTER VII.

HYDROSTATICS.

140. *Transmission of force through a fluid.*—A fluid transmits pressure equally in all directions (Art. 14). To bring out the meaning of this statement we will consider the following case:—Let B C D E be a vessel in which is inserted a small cylinder fitted with a piston A, and for distinctness we will suppose the area of A to be one square inch. The vessel is supposed to be filled with a fluid whose weight we will put out of the question. Suppose the piston to be pressed down with a force p, and to be at rest. Take areas of a square inch a-piece on the sides of the vessel, such as U, X, Y, Z, each of these areas sustains a pressure of p units in a perpendicular direction. So that if we take any one face, as C D or B C, and suppose its area to be 100 sq. in., the whole pressure on it will be 100 p, but distributed uniformly over it. We see, then, that the pressure transmitted through the fluid to any surface is a distributed pressure or stress, and must be estimated as a force of so many units per unit of area, e.g. of p units per square inch. In the next place, not only is pressure transmitted to the sides of the vessel, but every portion of the fluid mass is in a state of compression. Suppose a plane H K to be drawn across the vessel, the force p is transmitted through H K in undiminished

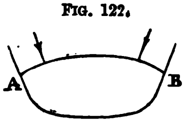
FIG. 121.



intensity, viz. at the rate of p units per square inch, and at right angles to H K, as indicated by the arrow-heads. These forces are balanced by equal reactions transmitted from the sides of the vessel through H K in the opposite direction. Again, let M be any portion of the fluid contained by an imaginary surface; let lines be drawn at right angles to this surface at every point; M will be subjected to pressure at every point of its surface along these lines at the rate of p units per square inch. The statement that fluids press equally in all directions, as now explained, can be shown to depend on the fact that fluids offer no sensible resistance to a tangential force,[1] a property enjoyed by all ordinary fluids when in a state of rest. The manner in which the bubble of a spirit-level instantaneously changes its position with the slightest change in the inclination of the instrument is, perhaps, the most conclusive experimental evidence that can be offered of the fact that fluids at rest offer no sensible resistance to tangential force.

141. *Form of surface of a fluid.*—Let A B C be any vessel of moderate size, containing a fluid whose surface is A B. The force on the surface A B must at each point act at right angles to the surface; for otherwise there would be a tangential force at one or more points, and this would cause motion. If the fluid is under the action of gravity, the force acts along parallel lines, and consequently A B will be a horizontal plane. If, however, A B were the surface of water in a large basin, such as a lake or inland sea—e.g. Lake Superior, or the Caspian Sea—the direction of gravity near A would not be parallel to its direction near B; for these directions converge very nearly to the centre of the earth, and the distance A B has an appreciable magnitude

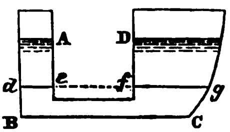
FIG. 122.



[1] Rankine's *Manual of Applied Mechanics*, p. 100.

in comparison with the radius of the earth. Consequently, in such cases the surface A B is sensibly curved when at rest. Such cases, however, are not contemplated in the following articles. We shall, in fact, consider that the fluid is contained in vessels of moderate size, and is under the action of gravity, unless the contrary is distinctly specified. What is true of the surface of the fluid in a single vessel is equally true of a fluid in two vessels between which there is a free communication. Thus, if A B and D C are two vessels containing a fluid, and connected by a tube B C through which the fluid can pass freely, the surfaces A and D will be in the same horizontal plane.

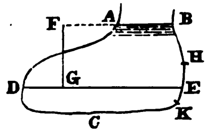
FIG. 123.



142. *Pressure on a horizontal section of a fluid.*—In fig. 123 suppose A $d\,e$ to be a cylinder with its axis vertical and section one square inch. As the sides of the vessel are vertical, no part of the weight of the fluid A $d\,e$ can be sustained by them, and consequently the whole is borne on $d\,e$. If, then, ρ is the density of the fluid, h the height A d and g the accelerative effect of gravity, the pressure on $d\,e$ is a force of $g\,\rho\,h$ units. Now let us produce the horizontal line $d\,e$ to $g\,f$; the fluid below this line would be at rest if all the superincumbent fluid were removed, and consequently it merely serves as a means of transmitting the pressure on $d\,e$ to $f\,g$. Hence the pressure per square inch on $f\,g$ *upward*, transmitted from A d, is $g\,\rho\,h$ units; and as this is balanced by the fluid in D $f\,g$, the pressure produced by that fluid downward on $f\,g$ must, therefore, be at the rate of $g\,\rho\,h$ units per square inch. Now this result is obtained irrespectively of the form of the vessel D $f\,g$; consequently, whatever be the form of the vessel, the pressure on each square unit of a horizontal section is $g\,\rho\,h$. Thus, if A B C is a vessel of

any shape filled with a fluid whose surface is A B, take any horizontal section of the vessel D E, and from G, any point in D E, draw the vertical line F G to meet B A produced in F; then the pressure on each square inch of D E is that due to a column of the liquid whose height is F G and cross section a square inch. Consequently the whole pressure on D E is that due to a column of the liquid whose base is D E and height F G.

FIG. 124.



Ex. 156.—Suppose the area of the section D E to be 4 sq. ft., and F G to be $1\frac{1}{2}$ ft., the fluid being water. The pressure on D E is that due to 6 cubic ft. of water, whether the actual quantity of water above D E be more or less than 6 cub. ft. It is frequently convenient to leave the answer in the above form; but it may be necessary to express the pressure in other units; and in a large number of cases it is sufficient to reckon the cubic foot of water as containing 1000 oz.; the pressure on D E would then be 375 lbs., i.e gravitation units. If greater exactness is required, the cubic inch of water may be taken to weigh 252·5 grains. But complete exactness cannot be attained without reference to temperature (see *note*, Art. 5). This assumes that the force of gravity is sensibly the same as in London. If the question is such that the variation of gravity from its force in London enters the question, the pressure should be expressed in absolute units, e. g. if the force of gravity were 32·1 and the cubic inch of water were exactly 252·5 grains, the pressure on the section D E would be $6 \times 1728 \times 252{\cdot}5 \times 32{\cdot}1 \div 7000$, or 12,005 absolute units.

Ex. 157.—In fig. 121 suppose A to be 2 ft. above C D, and the pressure on A to be 400 absolute units; suppose also that the fluid is water (each cubic foot containing 1000 oz.), and let the pressure on an area of 4 sq. in. forming part of C D be required. The pressure will be that transmitted from P together with that due to the weight of a column of water whose height is 2 ft., and cross section 4 sq. in. The former pressure is 4×400 absolute units, the latter $(2 \times \frac{1}{36} \times 1000 \div 16) \times g$ absolute units; the sum of these divided by 32·1912 will give the pressure in gravitation units. If we take g as approximately equal to 32, and also use 32 as a divisor instead of 32·1912, we obtain $53\frac{17}{36}$ lbs. as the approximate amount of the pressure.

143. *Equilibrium of two fluids in a tube.*—In fig. 123 let there be two fluids incapable of mixing, one in A B, the other in D C; let fg be the surface which

separates the two fluids; produce gf to de; the condition of equilibrium is that the heights of A and D above dfg be such that the pressures per square unit on de and fg produced by the two fluids be equal; in other words, if ρ and σ denote the densities of the fluids in Ad and Bg, we must have

$$\text{A}e \times \rho = \text{D}f \times \sigma.$$

In order that the fluids may continue in equilibrium, it is necessary that the fluid below be denser than that above fg, the surface which separates them. It is evident that fg will be a plane surface.

144. *Pressure at any point of a side of vessel containing fluid.*—Referring to fig. 124, we have seen that the fluid pressure throughout the section DE is $g\rho h$ per square inch; consequently, the pressure at the point E on the side of the vessel will be at the same rate, and exerted in the direction of the perpendicular at E. It varies, however, from point to point, being less at H and greater at K, in proportion to the depths of these points below the surface. It admits of proof that when any plane area is pressed by a fluid, the whole resultant pressure is found thus:—Let A denote the area pressed, and H the depth of the centre of gravity of the area below the surface; then the pressure is that due to a column of the fluid whose base is A and height H.

Ex. 158.—Suppose a cubical tank to have each edge 4 ft. long, and to be filled with water. The centre of gravity of each vertical face is 2 ft. below the surface, and consequently the fluid pressure on it is that due to 16×2, or 32 cub. ft. of water, or about 2000 lbs. (gravitation units). The pressure on the bottom is, of course, that due to 64 cub. ft. of water. So that the whole pressure on each side is half the pressure on the bottom.

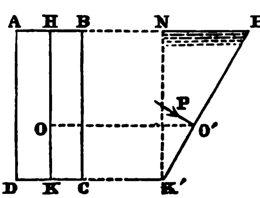
Ex. 159.—Suppose a cylindrical vessel to be filled with water, and placed with its axis horizontal; what will be the pressure on one end supposing its radius to be 3 ft?

The surface of the water is in this case at the highest line of the cylinder, and consequently the centre of gravity of the end is 3 ft. below the

surface. Also the area of one end is $\pi \times 3^2$ sq. ft. Consequently the pressure on the end is that due to $\pi \times 3^2 \times 3$ or 84·823 cub. ft. of water, i. e. about 5300 lbs. (gravitation units).

145. *Centre of fluid pressures.*—When a plane area is under the pressure of a fluid, the point in which the direction of the resultant pressure meets the area is called the centre of fluid pressures. The position of the centre depends on the size and form of the area, and on the position of the area in the fluid. There are methods by which its position can be found in all cases. Let A B C D be a rectangle sustaining the pressure of a fluid, and let the side A B be on the surface of the fluid; draw the line H K through the middle points of A B and C D respectively; take H O equal to $\frac{2}{3}$rds of H K; the point O is the centre of fluid pressures on the rectangle A B C D. This result is true, whatever be the inclination of the plane of A B C D to the vertical. Thus, if H′ K′ represents the line H K when the rectangle is seen edgewise, the centre of fluid pressures is at O′, H′ O′ being $\frac{2}{3}$rds of H′ K′. The fluid pressure (P) is, of course, exerted in a direction at right angles to H′ K′, and, if measurements are in feet, it equals that due to $\frac{1}{2}$ K′ N × A B C D cubic feet of fluid; the line K′ N being drawn vertically to meet the surface of the fluid in N.

FIG. 125.

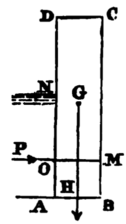


The following cases may also be noticed:—(*a*) If the area pressed is horizontal, the centre of fluid pressures coincides with the centre of gravity of the area; (*b*) in the case of any area whose dimensions are small compared with its depth below the surface, the centre of fluid pressures and the centre of gravity very nearly coincide, e.g. a circle 1 ft. in radius is placed in a fluid, with its *plane vertical*, and its centre 12 ft. below the surface, it

can be shown that the centre of fluid pressures is only $\frac{1}{4}$ in. below the centre of the circle. If the circle were only just covered by the fluid, the centre of the fluid pressures would be 3 in. below the centre of the circle.

Ex. 160.—Let ABCD be a wall of brickwork (112 lbs. per cubic foot) 12 ft. high and 3 ft. thick; the face AD sustains the pressure of water. Find the height AN to which the water may rise without overthrowing the wall; it being assumed that the wall is supported only by its own weight.

FIG. 126.



The forces acting are the weight of the wall, and the fluid pressure on AD; as the forces are the same on each foot of the length of the wall, we may make our calculations as if the wall were only 1 foot long. The weight of the wall (W) is therefore that of 12×3 cub. ft., or 4032 lbs. Draw a vertical line GH through G, the centre of gravity of ABCD, W will act along GH. The pressure of the water (P) on AN is that due to the weight of $AN \times AN$ cub. ft. of water, or $\frac{125}{4} AN^2$ lbs. If AO is taken equal to $\frac{1}{3}$ AN, P will act through O in a direction at right angles to AN; produce PO to cut BC in M. Now, if N is the extreme height to which the water can rise, the forces P and W will be in equilibrium on B, as if B were a fulcrum, so that (Art. 45) $BH \cdot W = BM \cdot P$, i. e.

$$\tfrac{1}{3} AN \times \frac{125}{4} AN^2 = \tfrac{3}{2} \times 4032;$$

Therefore, $$AN^3 = \frac{18 \times 4032}{125},$$

or $$AN = 8{\cdot}34 \text{ ft.}$$

If we suppose everything to be the same, except that the height of the wall is determined by the condition, that the wall will just sustain the pressure when the water rises to the top, we must reason thus:—In this case the weight of the wall (supposed to be 1 ft. long) is 336 AD lbs., while the pressure of the water when on a level with the top of the wall is $\frac{125}{4} AD^2$ lbs.;

consequently $$\frac{125}{12} AD^3 = 504\, AD;$$

therefore, $$AD = 6{\cdot}96 \text{ ft.}$$

Ex. 161.—A rectangular box whose base is a foot square and height 12 ft. is filled with water; a door 4 in. deep and 1 ft. wide is made in one face, the centre of the door being 5 ft. below the surface. If the door is

opened, so that the water rushes out, what effect will it have on the box? Consider an area equal to that of the door on the opposite side of the vessel; before the door is opened the pressure of the water on the area is balanced by the pressure of the water on the door. When the door is opened, the pressure on it is removed, is, in fact, expended in causing the outflow, so that the pressure on the equal opposite area is no longer balanced, and consequently tends to overthrow the vessel in a direction opposite to that of the issuing stream. To ascertain whether this tendency will be sufficient to overthrow the vessel, we may proceed thus:—The pressure on the area is that due to $\frac{1}{3} \times 5$ cub. ft. of water, and we may consider the centre of pressure to coincide with the centre of gravity of the area; consequently the moment of the force with regard to the edge of the vessel round which motion tends to take place is $\frac{1}{3} \times 5 \times 7$. The weight of the fluid is that of 12 cub. ft. of water, and its moment with regard to the same edge is $\frac{1}{2} \times 12$. We see that this is very much smaller than the former moment, and consequently the opening of the door will overthrow the vessel, unless the weight of the vessel itself is very great.

Ex. 162.—In the last case, if x were the depth of the centre of the door below the surface of the water, the pressure on the area opposite to the door would be the weight of $\frac{1}{3}x$ cub. ft. of water, and the moment this pressure with respect to the edge round which it tends to make the body turn is $\frac{1}{3}x(12-x)$. If this is made equal to the moment of the weight of the water, we have

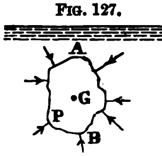
$$\tfrac{1}{3}x(12-x)=6;$$

therefore, $$x=6\pm 3\sqrt{2}=6\pm 4{\cdot}242.$$

Hence, if the middle of the door is placed either $1\frac{3}{4}$ or $10\frac{1}{4}$ ft. below the surface, the pressure of the water will be just sufficient to overthrow the box; in any intermediate position the pressure would be more than just sufficient, assuming the vessel to be without weight.

146. *Pressure on a body immersed in a fluid.*—Consider any portion A B of the fluid inclosed by an imaginary boundary. A certain pressure will be exerted by the surrounding fluid at each point of A B; the amount and direction of the pressure at any one point, say P, will depend on the depth of P below the surface of the fluid, and on the form of the boundary at P. Now, as the fluid is at rest, these pressures must exactly support the weight of the enclosed fluid, i.e. of A B, and consequently their resultant

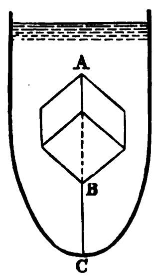
FIG. 127.



R must equal the weight of A B, and must act vertically upward through its centre of gravity G. Now suppose any solid whatever to fill exactly the space previously occupied by A B; the fluid pressures at the different points of the boundary will be unchanged; consequently their resultant will be the same as before; in other words, it will equal the weight of the displaced fluid, and will act vertically upward through the centre of gravity of the displaced fluid, i.e. through the point G, determined when the space A B is filled with fluid. If both fluid and body are of uniform density, the centre of gravity of the fluid displaced coincides with that of the body.

Ex. 163.—Let A B be a cube of elm, whose edge is 1 ft. long, fastened by the angle B to a string B C, the end of which is tied firmly to C, a point in the bottom of the vessel. If the cube is wholly immersed, it displaces about 1000 oz. of water, while its own weight is about 600 oz. Consequently the cube is urged upward by a force of about 400 oz. acting through its centre of gravity. As this force is balanced by the reaction of the bottom of the vessel transmitted through the thread, the cube will adjust itself in such a position that the diagonal A B and the string B C shall be in the same vertical line. This example suggests the following case:— Suppose the vessel containing the cube to be put into one pan of a pair of scales, and counterpoised by weights in the other pan. The counterpoise will equal the united weights of vessel, water, and wood. The excess of the upward pressure of the water above the weight of the wood (400 oz.) against the cube causes an equal pressure on the bottom of the vessel, but they are in equilibrium, being transmitted in opposite directions through the string; consequently they have no effect on the counterpoising weights. Now, suppose the string to be cut. So long as the cube is under water, it is being forced up by the pressure of 400 oz. The reaction on the bottom is no longer balanced, and the scale-pan will begin to descend. The force causing this descent will diminish as the cube rises out of the water, and become zero when it floats on the water, and thus the state of equilibrium will be restored without change of weights.

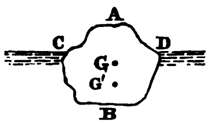
FIG. 128.



147. *Equilibrium of a floating body.*—Let A B be a body floating in a fluid, let G be its centre of gravity, and

let W denote its weight. We know by the last article that the resultant of the fluid pressures on the immersed part CDB equals the weight of the displaced fluid, and acts vertically upward through (G′) the centre of gravity of the displaced fluid. But, as the fluid pressures support the body, their resultant must equal the weight of the body, and act vertically upward through its centre of gravity. Hence a floating body must fulfil the following conditions:—(*a*) The weight of the body must equal the weight of the fluid displaced; (*b*) the centre of gravity of the body and that of the fluid displaced must be in the same vertical line.

FIG. 129.



Ex. 164.—A cube of larch-wood (sp. gr. 0·6) will float in water with an edge vertical, and in such a position that 6–10ths of the edge are in water.

Ex. 165.—A cube of cast-iron, whose edge is 1 in., floats in mercury. When water is poured into the vessel, so as to cover the cube, what change is produced in the position of the cube?

If x is the portion of the edge immersed in mercury, we must have

(Table of sp. gr.) $$1 \times 7{\cdot}2 = x \times 13{\cdot}5;$$

so that $\frac{72}{135}$, or $\frac{8}{15}$ of an inch are immersed. Suppose water to be poured on, and that y is the portion of edge now in the mercury, so that the quantity of mercury displaced is y (cubic inch) and of water $1-y$ cub. in.; the joint weight of these is the same as that of a cubic inch of iron. Hence,

$$y \times 13{\cdot}5 + (1-y) = 1 \times 7{\cdot}2.$$

or y equals $\frac{62}{125}$ of an inch. Now, as $x-y=\frac{8}{15}-\frac{62}{125}=\frac{14}{375}$ in., we see that the cube rises in the mercury, being in part supported by the water.

148. *Applications to determination of specific gravities, &c.*—If we know the weight of a body w, and the weight of an equal volume of water w_1, the specific gravity of a body is $w \div w_1$ (Art. 5). The former quantity is ascertained directly by weighing, the latter is *inferred* by means of the principle of Art. 147, as will appear in sequel.

1. *Solid heavier than water.*—Let the weight of the body in air be W, and its weight in water w, then the weight of an equal volume of water is $W-w$, and consequently the specific gravity is $W \div (W-w)$. Thus, a piece of iron weighs 1035 gr. in air, and 885 gr. in water; the weight of the displaced water is $1035-885$, or 150 gr.; and consequently the specific gravity is $1035 \div 150$, or 6·9.

For making the necessary weighings, the balance may be arranged thus:—One of the pans (A) has a hook on its under side, and is suspended from the beam by shorter threads than the other pan (B). The body is first placed in A and weighed; it is then fastened by a thread to the hook and placed in a cup of water, so as to be wholly covered by the water, and not to touch the bottom of the vessel. The weights in B will now preponderate. Weights can be placed in A till the equipoise is exactly restored. These give the weight of the displaced water, viz. $(W-w)$.

2. *Solid lighter than water.*—Fasten to the body a *sinker*, e.g. a piece of brass sufficiently heavy to bring the body completely under water. Ascertain the following weights:—W, the weight of the body in air; W_1, the joint weight of the bodies in water, and S the weight of the sinker in water; also let x denote the weight of the sinker in air, and w the weight of the water displaced by the body. Now the joint weight of the bodies in air $(W+x)$ equals their joint weight in water (W_1), together with the weight of water displaced by sinker $(x-S)$, and the weight of water displaced by the body (w), or

$$W+x=W_1+x-S+w;$$

therefore $$w=W+S-W_1,$$

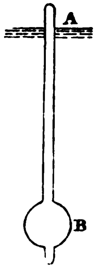
and the specific gravity required is $W \div w$. Thus, a piece of cork weighs 200 gr.; when fastened to a sinker their joint weight in water is 450 gr.; the weight of the sinker in water is 1030 gr. Hence weight of water displaced by cork is 780 gr., and the specific gravity of the cork is $200 \div 780$, or 0·256.

3. *Specific gravity of a fluid.*—Take a ball of glass or platinum and find its loss of weight in water (W), then its loss of weight in the fluid (W_1). As these are respectively the weights of equal volumes of water and the fluid, the required specific gravity is $W_1 \div W$. There are other methods of finding the specific gravity of a fluid; such are the two that follow.

4. *The specific gravity bottle* is merely a small glass bottle fitted with a glass stopper very accurately ground, so that if the stopper is taken out and put in again several times it always returns to exactly the same place. A fine bore is made through the length of the stopper. Fill the bottle and let the fluid rise in the neck into the part which the stopper will fill; on putting in the stopper the surplus of fluid will escape through the bore. If the same process is repeated with several fluids, their volumes will be exactly equal. Now suppose the empty bottle to be exactly counterpoised by a piece of lead or by shot. Let this counterpoise be placed in one scale-pan and the bottle full of water in the other, and suppose the additional

weight required to bring the whole to a balance to be 1035 gr. The water in the bottle weighs 1035 gr. Let the bottle now be filled with another liquid (say alcohol), and suppose it to be exactly balanced by the counterpoise and 823 gr.; the weight of the alcohol is therefore 823 gr. Hence, as the volumes are equal, the specific gravity of the alcohol is $823 \div 1035$, or 0·795.

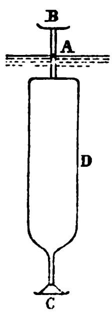
FIG. 130.



5. *The hydrometer* consists of a glass tube stopped at one end, and having a globe at the other, in which so much mercury is placed, that when the instrument floats in a liquid the stem is vertical. As the weight of the liquid displaced always equals the weight of the instrument, the depth to which the stem is immersed supplies the means of inferring the specific gravity, and if the stem is properly graduated, the specific gravity can be ascertained by inspection.

6. *Nicholson's hydrometer* consists of a small pan B, joined by a stem to a hollow cylinder D, below which is placed another small pan C. The instrument is loaded so that it floats with its axis vertical. On the stem is a marked point A. It is known that some standard weight (say 1000 gr.) will sink the instrument to A in water. (*a*) It can be used for weighing a small body, thus:—Let such a body be placed in B, and suppose that 250 gr. must also be placed in B, to sink the instrument to A, it is plain that the weight of the body is $1000-250$, or 750 gr. (*b*) It can be used for determining the specific gravity of a solid, thus:—Suppose the body (considered above) to be placed in C and that now 350 gr. must be placed in B to sink the instrument to A, the weight of the body in water is plainly $1000-350$, or 650 gr., so that the weight of the equal volume of water is 100 gr., and the specific gravity of the body is $750 \div 100$, or 7·5. (*c*) It can be used for finding the specific gravity of a fluid, thus:—Suppose the instrument placed in a fluid and that a weight of 930 gr. is required to sink it to A, the weights of equal volumes of the liquid and water being 930 and 1000 gr. respectively, the specific gravity of the liquid must be 0·93.

FIG. 131.



7. *Determination of volume.*—Suppose the body to weigh w gr. in air, and w_1 in water, the weight of an equal volume of water is $w-w_1$; this divided by 252·5 gives the volume of the body in cubic inches, e.g. a piece of copper wire 3 in. long weighs 44 gr. in air and 39 in water; what is its diameter? Its volume is $5 \div 252{\cdot}5$, or $\frac{2}{101}$ cub. in., the area of its cross section is $\frac{2}{303}$ sq. in., and hence its diameter is 0·0917 of an inch.

8. *Correction for buoyancy of air.*—In the above articles we have spoken of the weight of bodies in water; when great exactness is necessary this must be understood to mean that the weight of the body is compared with

that of an equal volume of distilled water at some assigned temperature, e.g. 60°F. or 4°C. We have also spoken of the weight of the bodies in air; when great exactness is necessary both the body and the weights ought to be in vacuo. If the apparent weight of a body in air is W, we may ascertain (approximately) its weight in vacuo, thus:—Suppose m is the weight of the air displaced by the body, and w that displaced by the weights, the weight of the body, if taken in vacuo with the weights also in vacuo, would be $W+m-w$. Now at temperature 60° F. and under a pressure of 30 in. of mercury, water is 817 times as heavy as dry air;[1] consequently if S and s are the specific gravities of the body and of the weights (which need not be known with extreme exactness) we shall have 817 S m equal the weight of the body, and 817 $s\,w$ the weight of the weights, and each equal to W. Therefore (very nearly)

$$m=\frac{W}{817\,S} \qquad \text{and} \qquad w=\frac{W}{817\,s}.$$

Thus, a body weighs in air 4000 gr.; the specific gravity of the body is 1·6 and of the weights 8·4. We have $m=4000 \div 1307=3{\cdot}06$, and similarly $w=0{\cdot}58$; consequently the corrected weight is 4002·48 gr. A result sufficiently correct for almost every purpose, even supposing that the specific gravities (1·6 and 8·4) are determined only approximately.

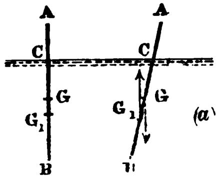
149. *Stability of flotation.*—Conceive a body to be placed in a fluid in such a manner that the conditions of equilibrium stated in Art. 147 are fulfilled. It is a subject for further inquiry whether or not that position is one of stable equilibrium (Art. 25). To ascertain this point, suppose the body to be very slightly displaced; if, in this new position, the forces tend to make the body move further from its original position, that position was one of unstable equilibrium; if, on the other hand, the forces tend to restore the body to its original position, that position was one of stable equilibrium (at all events with reference to that particular direction of displacement).

Ex. 166.—Let AB be a long thin rod of uniform density placed in a vertical position in the water, CB being the part immersed, and suppose CB

[1] At 60° F. a cubic inch of water weighs 252·769 gr.; at 60° F. and barometer at 30 in. in London, 100 cub. in. of dry air weigh 30·940 gr. —Stewart, *On Heat*, p. 70-1.

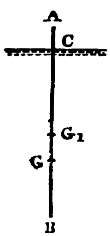
to be so long that the weight of the water displaced by C B equals the weight of the rod; take G the middle point of A B, and G_1 that of C B; these are the centres of gravity of the body and of the water displaced, and they are in the same vertical line. The conditions of Art. 147 are therefore both fulfilled, and the position is one of equilibrium. To ascertain whether the equilibrium is stable or not, suppose A B to be slightly inclined to the horizon as in fig. 132 (a); the relative positions of G and G_1 undergo no sensible change, and the rod is now acted on by two forces as shown in the figure (viz., the weight of the rod acting downward through G, and the fluid pressure acting upward through G_1). The tendency of these forces is to turn the rod round C, bringing B up and A down to the surface of the water. Accordingly A B's position (fig. 132) is one of unstable equilibrium.

FIG. 132.



If now we suppose a heavy point to be fastened to the end B, the rod will have less of its length above water, and its centre of gravity (G) will be nearer B than A. If G_1 is the middle point of B C, and if G is below G_1, it can be shown as in the last paragraph that the rod will be in stable equilibrium. If we denote by s the specific gravity, by W_1 the weight of the rod, and by W the weight of the *point* fastened to B, it can be easily shown that W must be less than $W_1\left(\frac{1}{s}-1\right)$ or the rod would not float, and greater than $W_1\left(\frac{1}{\sqrt{s}}-1\right)$ or the rod would not float vertically. The student should prove this, bearing in mind W is supposed to be a *point*. Thus, if the rod weighs 100 gr. and has a specific gravity of 0·81, it will float with its axis vertical, when the added weight is between 23 and 11 gr.[1]

FIG. 133.

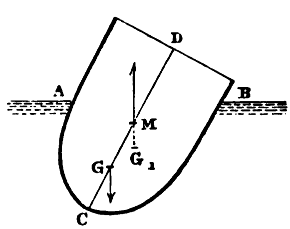


If we suppose A B (fig. 132) to be a thin plate seen edgewise, the same reasoning will apply, and the plate is in unstable equilibrium. We have therefore, in Ex. 166, an explanation of the well-known facts, that a rod commonly floats lengthwise, that a plate floats flat on the water, but that, if properly laden, a rod or a plate may float vertically. Other ordinary cases of stability have to be treated with reference to the *metacentre*, a point defined in the following article.

[1] If the added weight were less than 11 gr., the position of stable equilibrium would be one in which A B is inclined to the vertical; that position cannot be determined without taking account of the form and magnitude of the cross section of A B, and the question becomes one of some difficulty.

150. *The metacentre.*—Let ACB be a floating body very slightly displaced from its position of equilibrium; let G be its centre of gravity, and suppose the line CGD to have been vertical before displacement. Let G_1 be the centre of gravity of the fluid displaced, draw the vertical line G_1 M, cutting CD in M; on the supposition that the angle CMG_1 is exceedingly small, M is the *metacentre.* The equilibrium of the body before displacement was *stable* if the metacentre is above the centre of gravity, *unstable* if below. The figure shows the case in which M is above G; in this case the forces act as shown in the figure, and manifestly tend to replace CD in a vertical position.

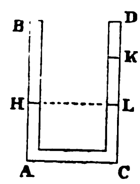
FIG. 134.



Ex. 167.—If we suppose the curve ACB (fig. 134) to be an arc of a circle, the direction of the fluid pressure at each point passes through the centre of the circle, consequently the metacentre of a sphere is at its centre. Also when a cylinder floats with its axis horizontal, and is displaced round the axis, the metacentre is at the centre of the cross section. In other cases the determination of the *metacentre* presents greater difficulties.

151. *Rotatory motion of a fluid.*—The case we shall consider is that of a fluid under the action of gravity in a vessel revolving with a uniform angular velocity (θ) about a vertical axis. The case has this peculiarity, that after the motion has been established, the particles of fluid are relatively at rest.

FIG. 135.



a. Let BACD be a bent tube of uniform bore, the legs of which, AB and CD, are at right angles to AC. There is a fluid in the tube, and it is supposed that the revolution takes place round the line AB. When the fluid ceases to move within the tube, its surfaces H and K will not be on a level. Through H draw the horizontal line HL. The question is: with a given angular velocity, what is the difference of level, KL? If we denote the cross section of the tube by A

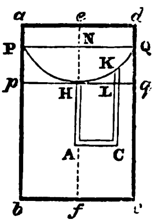
and the density of the fluid by ρ, the centrifugal force of the fluid in AC will be $\frac{1}{2}\rho A \theta^2 \times AC^2$ (Art. 122), or $\frac{1}{2}\rho A \theta^2 \times HL^2$. Consequently, in order that the fluid may be at rest in AC, a pressure must be exerted on it in the direction C to A equal to the centrifugal force. Now as the weights of the fluids HA and LC are in equilibrium, this force must be exerted by the weight of the fluid column KL or $g\rho A \times KL$. Hence we have

$$\theta^2 \times HL^2 = 2g \times KL;$$

or the velocity of the point L is the same as would be acquired by a point in falling through the height KL.

b. Suppose the vessel to be a cylinder ($abcd$) and to revolve round its axis (ef), the surface of the fluid will be concave, and the question is: what is the form of the surface? The figure represents a section of the vessel made by a plane passing through the axis, and PHQ the section of the concave surface of the fluid. Conceive a portion of the fluid to be contained in a tube HACK, the condition of relative equilibrium of the fluid within this tube will be the same as in the last paragraph. Draw the horizontal line pHg, cutting KC in L—the curve PQ must be such that at any point K the following relation holds good:—

FIG. 136.

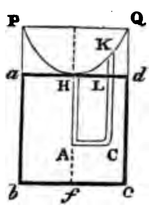


$$\theta^2 \times HL^2 = 2g \times KL.$$

This relation, as is well known, proves the curve PHQ to be a parabola, with its vertex at H, and axis coinciding with ef. The surface of the fluid is therefore that generated by the revolution of this curve round its axis. Join PQ cutting the axis N. It is a well-known property of the surface that the volume of the space PNQH is half the cylindrical space PQqp. Hence the quantity of water above pq is as much as would half fill the space PQqp.

c. Let $abcd$ be a cylinder with its axis Hf vertical, and suppose the cylinder to be exactly filled with a fluid, and then to be closed at top. The cylinder is now caused to revolve round its axis. We know that if the top were not closed some of the fluid would be thrown out, consequently a pressure will be exerted by the fluid against the top, tending to force it up. The question is: what is the amount of this pressure? If we suppose a part HACL of the fluid to be inclosed in a tube, as before, and produce CL to K, making

FIG. 137.



$$\theta^2 \times HL^2 = 2g \times KL,$$

the reaction of the top at L must equal the weight of KL. The same would be true of other points. Hence if PHQ is the same curve as that drawn in the last article, the whole pressure must equal the weight of a volume

of water that would half fill the cylinder P a d Q, i.e. the pressure must equal

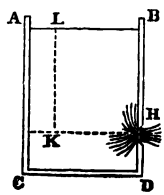
$$\tfrac{1}{2} g \rho \pi \,.\, \mathrm{H} d^2 \times d\,\mathrm{Q}.$$

Now $$\theta^2 \times \mathrm{H} d^2 = g \times d\,\mathrm{Q},$$

hence the required pressure equals $\frac{1}{2} \pi \rho \theta^2 \times \mathrm{H} d^4$.

152. *Motion of water through a small hole in the side of a vessel.*—Let A B C D be a vessel containing water, let a small hole H be made in one of its sides, and let the depth of the middle point of the hole below the surface of the water, viz. L K, be denoted by h; this height is commonly called the head of water. The theoretical velocity of efflux can be determined thus:—Consider a portion of the fluid of exceedingly small thickness t, which at any instant fills the hole; while it passes through a distance equal to t, it is acted on by a force equal to the pressure due to the head of water. If, then, A denotes the area of the hole, and ρ the density of the water, we have a mass $\rho \mathrm{A} t$ moving through a distance t under the action of a force $g \rho \mathrm{A} h$. The work done by the force is $g \rho \mathrm{A} h t$, and must equal the energy of the mass, viz. $\frac{1}{2} \rho \mathrm{A} t v^2$, where v denotes the velocity of efflux.

FIG. 138.



Hence $$v^2 = 2 g h.$$

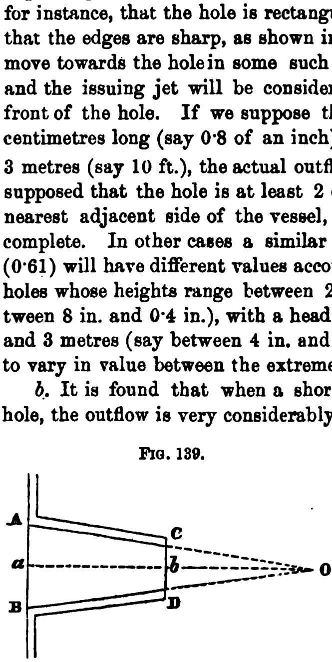
In other words, the theoretical velocity of efflux is that due to the *head* of water: a result commonly known as Torricelli's Theorem. The same result would be obtained if we suppose the hole to be made in the bottom of the vessel.

The quantity A v or $\mathrm{A}\sqrt{2gh}$ is called the *theoretical outflow*. It is the quantity of water which would flow through an area A in one second if every particle moved at right angles to the area with a constant velocity v.

a. The results obtained in the last article may be regarded as approximate values of the actual velocity and outflow. The correct values will depend upon the size of the hole, and the thickness of the wall. Suppose, for instance, that the hole is rectangular, with two of its sides vertical, and that the edges are sharp, as shown in the section in fig. 138. The fluid will move towards the hole in some such manner as that indicated in the figure, and the issuing jet will be considerably contracted at a short distance in front of the hole. If we suppose the hole to be a square with its sides 2 centimetres long (say 0·8 of an inch), and the head of water to be about 3 metres (say 10 ft.), the actual outflow is found to be $0{\cdot}61\,A\sqrt{2gh}$. It is supposed that the hole is at least 2 or 3 in. from the bottom and from the nearest adjacent side of the vessel, in order that the contraction may be complete. In other cases a similar formula will apply, but the coefficient (0·61) will have different values according to circumstances. For rectangular holes whose heights range between 2 decimetres and 1 centimetre (say between 8 in. and 0·4 in.), with a head of water ranging between 1 decimetre and 3 metres (say between 4 in. and 10 ft.), the coefficient has been found to vary in value between the extreme values 0·592 and 0·666.[1]

b. It is found that when a short pipe or *ajutage* is inserted into the hole, the outflow is very considerably increased. Let A B D C be such a pipe, whose form is a truncated cone. The area of the section C D is A, and the length *a b* is 2·6 times C D. In this case, when the angle of the cone is 13° 24′, it has been found that the actual outflow is given by the formula $0{\cdot}946\,A\sqrt{2gh}$. When *ajutages* were tried, differing from the above only in the size of the angle O, the coefficient (0·946) was found to change its value.

FIG. 139.



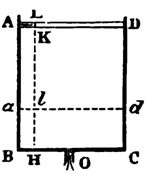
Thus, when the angle A O B was zero, i. e. when the ajutage was a cylinder, the actual outflow was found to be $0{\cdot}829\,A\sqrt{2gh}$. When the angle was increased to 48° 50′, the actual outflow was $0{\cdot}847 \times A\sqrt{2gh}$. And of all the angles tried 13° 24′ was found to give the greatest actual outflow. The velocity of the outflow was found not to follow the same rule, but to increase as the angle A O B was increased, being $0{\cdot}830\sqrt{2gh}$ when the ajutage was cylindrical, and $0{\cdot}984\sqrt{2gh}$ when the angle of the cone was 48° 50′.[2] It will be seen that the last velocity very nearly equals the theoretical velocity of outflow. Ajutages of other form have been made the subjects of experiment, but the results obtained need not be given here.

[1] Morin, *Aide Mémoire*, p. 13. Experiments due to MM. Poncelet and Lesbros.

[2] Morin, *Aide Mémoire*, p. 28. Experiments due to M. Castel.

153. *Time of emptying a cylindrical vessel.*—Let A B C D be a vessel filled with water up to the level A D, and suppose a hole, O, to be made in its bottom; if the hole is small in comparison with the cross section of the vessel, the surface of the fluid will descend very nearly in parallel planes, and the velocity of the descent can thus be determined. Draw the vertical line H L, and take L K a very small part of H L, and suppose t is the short time in which the surface falls through the distance L K. If A denotes the area of the cross section A D, the outflow in the time t will be A $\times$ L K. If h denote the height H L, and a the effective area of the hole, the outflow in the time t will be $t\,a\sqrt{2\,g\,h}$, since the velocity of the outflow does not sensibly change.

FIG. 140.



Hence,
$$\text{A}\times\text{L K}=t\,a\sqrt{2\,g\,h}.$$

We see, then, that the area describes the distance L K in the short time t; consequently the velocity of its descent (v) will equal L K $\div t$.

Therefore,
$$\text{A}\,v=a\sqrt{2\,g\,h}.$$

This equation will be true whatever the form of the vessel. We will now assume the vessel to be cylindrical. When the surface has fallen to $a\,d$, let the velocity of its descent be denoted by v_1, and H l by h_1, and we have

$$\text{A}\,v_1=a\sqrt{2\,g\,h_1}.$$

Hence,
$$v_1^2=v^2-\frac{2\,g\,a^2}{\text{A}^2}(h-h_1);$$

or,
$$v_1^2=v^2-\frac{2\,g\,a^2}{\text{A}^2}\cdot\text{L}\,l.$$

Comparing this result with Art. 89, Eq. 9, we see that

the velocity of the descending surface is uniformly retarded, and that the rate of retardation is $\frac{g\,a^2}{A^2}$.

Assuming the result arrived at to be exactly true, the time of emptying the vessel (as is plain from Art. 89) will be given by the equation

$$H\,L = \frac{1}{2} \cdot \frac{g\,a^2}{A^2} \cdot t^2.$$

a. The following conclusion will also follow:—Take two equal cylindrical vessels A and B, and let them be filled with water up to the same height. Suppose they have equal small holes in their bottoms, but suppose the water is allowed to run out of A, while additional water is continually poured into B, so that the level of the water in it is maintained constantly at the original height. Then during the time in which A becomes empty, the outflow from B will be double that from A. The student should prove this.

QUESTIONS.

1. Describe briefly how pressure is transmitted through a fluid contained in a closed vessel.

2. In fig. 121, if A were 1 sq. in.; p, a force of 5 lbs., and M a cube, each edge of which is 3 in., to what pressures would the cube be subjected? (Weight of fluid not to be considered.)

3. How is the pressure of a fluid estimated at any point? How does it appear that a fluid mass offers no sensible tangential resistance?

4. When the surface of a fluid is free to move, what consideration determines its shape? How does this apply to the surface of a fluid contained in a vessel of moderate dimensions under the action of gravity?

5. State the rule for determining the pressure of a fluid on any horizontal section of its mass.

6. At a depth of 80 ft. of water, what is the pressure [1] per square inch?
Ans. $34\frac{13}{18}$ lbs.

7. A can is closed at top; through the lid goes a pipe which reaches to a height of 50 ft.; the lid has an area of 49 sq. in., and the pipe a cross section of 1 sq. in.; suppose the can and pipe to be full of water, what is the force tending to burst open the lid? Does the cross-section of the pipe affect the answer?
Ans. $1041\frac{2}{3}$ lbs.

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[1] In the following question a cubic foot of water is taken to weigh 1000 *oz.*, unless the contrary is expressed.

8. In the last question suppose the can to be 2 ft. high, and to be of a cylindrical form; show that the downward pressure on to the bottom exceeds the upward pressure against the lid by the weight of the water.

9. When two fluids which do not mix are put into a tube, what is their position of equilibrium?

10. Take a tube with a uniform cross section of 1 sq. in.; let it consist of a horizontal part 6 in. long, and two vertical legs (A and B); when 12 cub. in. of mercury are poured into the tube, there will be (of course) 6 cub. in. in the horizontal part, 3 in A and 3 in B; if 45 cub. in. of water are poured gradually into A, show that there will be $1\frac{1}{3}$ in. of mercury in A and $4\frac{2}{3}$ in B. What will happen if 36 more cubic inches of water are gradually poured into A? and what if 36 more?

11. State the rule for finding the magnitude of the resultant of the fluid pressure on any given plane area.

12. A cylinder, the radius of whose ends is 2 ft. long, lies lengthwise at the bottom of a tank 10 ft. deep; what is the pressure of the water on each end? *Ans.* 6283 lbs.

13. A cube, one of whose edges is 2 ft. long, is filled with water and closed; it is turned round one horizontal edge till the opposite edge is vertically over it (so that four faces are inclined to the horizon at an angle of 45°): find the pressure of the water on the faces of the cube.

Ans. 176·7 lbs.; 353·6 lbs.; 530·3 lbs.

14. What is meant by the centre of fluid pressures? When a rectangle has one edge on the surface of the fluid, how is the centre of the fluid pressure on one face of it determined?

15. A rectangle 1 ft. wide and 3 ft. long is placed in water with one edge on the surface, and the parallel edge 2 ft. below the surface; define completely the resultant of the fluid pressures on either face of the rectangle.

16. Mention circumstances under which the centre of fluid pressures coincides with, and others under which it nearly coincides with the centre of gravity of the area pressed.

17. Let ABCD be a rectangle divided into two equal rectangles by a line EF drawn parallel to AB; let AB be on the surface of water and BC vertically downward; also let AB and BC be 2 ft. and 6 ft. long respectively; find the pressure of the water on ABCD and ABEF; and hence find the centre of the fluid pressure on EFCD. *Ans.* $4\frac{2}{3}$ ft. below AB.

18. A reservoir is divided into two parts by a brick wall 12 ft. high and 3 ft. thick (112 lbs. per cubic foot); the water on one side rises to the top, but is at a lower level on the other side; at what difference of level will the wall become unsafe? *Ans.* 1·53 ft.

19. In Ex. 161, if the door is opened 6 ft. below the surface of the

water, what must be its height (its width being 1 ft.) if the vessel jus stands? *Ans.* 2 in.

20. When a body is wholly immersed in a fluid, what is the magnitud and direction of the resultant of the fluid pressures?

21. If the cylinder in Q. 12 were 8 ft. long, what would be the magni tude of the resultant of all the fluid pressures on it? If it were of beec wood, what would be the whole force urging it upward? Supposing it move as a point moves, how long would it be before beginning to emerge ($g \doteq 32$)? Supposing it to be tied to the bottom of the tank by a string 1 ft. long fastened to the centre of one end, in what position would it come to rest, and what would be the fluid pressure on each end?
Ans. (1) 6283 lbs.; (2) 1885 lbs.; (3) 0·94 sec.; (4) 785 lbs., 7069 lbs.

22. What conditions must be fulfilled if a body float?

23. A rectangular vessel, open at top—like a barge—made of iron plates ½ in. thick, is 30 ft. long, 12 ft. wide, and 6 ft. deep; what load will just sink it? *Ans.* 117,450 lbs.

24. The area of the cross section of a ship at the water-line is 9000 sq. ft.; what additional load will sink it 3 in. *Ans.* 140,625 lbs.

25. A rod of beech-wood (A B) 12 ft. long, with a small uniform cross section floats on the surface of still water; by means of a thread tied to it, the end (A) is lifted to a moderate height out of water, so that the rod floats obliquely: find how much of the rod will be immersed. Why must the thread take a vertical position? *Ans.* 5·43 ft.

26. State the method of finding the specific gravity of an insoluble solid heavier than water. Describe briefly the specific gravity balance.

27. A body weighs 2570 gr. in air and 1640 gr. in water; what is its specific gravity? *Ans.* 2·76.

28. How is the specific gravity of a solid lighter than water found?

29. A body weighs 590 gr. in air; the sinker weighs 750 gr. in water; body and sinker weigh 420 gr. in water; what is the specific gravity of the body? *Ans.* 0·64.

30. How can the specific gravity of a fluid be found by the balance?

31. A ball of glass weighs 3000 gr. in air, 2020 gr. in water, and 2060 gr. in wine; what is the specific gravity of the wine?
Ans. 0·959.

32. Describe briefly:—(1) The specific gravity bottle, mentioning the precautions taken for making successive measures equal in volume; (2) the common hydrometer; (3) Nicholson's hydrometer, explaining how this instrument can be used for finding the weight and specific gravity of a small body, and the specific gravity of a fluid.

33. In a common hydrometer whose stem is of uniform section, it is found that the instrument sinks to a point A in water, to B in a fluid whose specific gravity is 0·95, and to C in a fluid whose specific gravity is 0·97; what is the ratio of AB : AC? *Ans.* 97 : 57.

34. The standard weight of a Nicholson's hydrometer being 2000 gr., it is found that when a small body is in the pan B (fig. 131) 1100 gr. are needed to sink the instrument to the mark A; but when the body is at C, 50 more grains are required to sink it to A; what is the specific gravity of the body? *Ans.* 18.

35. State a simple method of determining the volume of a small body heavier than water. Assuming that a cubic inch of water weighs 252·5 gr., what is the volume of a body which weighs $50\frac{1}{2}$ gr. less in water than in air? *Ans.* 0·2 cub. in.

36. The specific gravity of a body is said to be obtained by dividing the weight of the body in air by its loss of weight in water; in what respect is the statement wanting in exactness?

37. If W is the apparent weight of a body in air, and m and w the weights of the air displaced by the body and the weights, what is the true weight of the body?

38. The apparent weight of a substance (body and weights in air) is 5040 gr.; the approximate specific gravities of bodies and weights are 0·75 and 8·4; determine a close approximation to the true weight of the body. *Ans.* 5047·5 gr.

39. By what test can it be ascertained that a position in which a floating body fulfils the conditions of equilibrium would also be one of stability?

40. A common wine cork is nearly 2 in. long; cut from it a slice (like a wafer) about a quarter of an inch thick; if placed in water, the cork and the slice are observed to float in different positions; explain the observed difference.

41. A rod weighs 12 oz., and has a specific gravity of 0·64; what must be the weight of a heavy point fastened to one end that will make it float vertically? *Ans.* Between 3 oz. and $6\frac{3}{4}$ oz.

42. When an uncorked bottle is put into water, why does it right itself and float nearly vertically when partly filled with water?

43. Define the *metacentre*. When the position of the metacentre is known, how can the stability or instability of the flotation be inferred? Mention some cases in which the position of the metacentre can be easily found.

44. A sphere, loaded by a heavy point being placed anywhere within it—not at its centre, floats; what will be its position of stable equilibrium?

45. When a pontoon or cask is afloat, why is it difficult or impossible for a man to sit astride it?

46. A tube with its legs vertical rotates uniformly about an axis coinciding with one of them; there is water in the tube; find the difference between the level of the water in the legs.

47. If the legs are 3 ft. apart horizontally, and the tube makes 90 turns a minute; what is the difference between the levels ($g=32$)?

Ans. 12·5 ft.

48. In the last question, what would have been the result had g been equal to 24? Why should g affect the answer? *Ans.* 16·7 ft.

49. A cylinder with water in it revolves uniformly round its axis, which is vertical; how can the form of the surface be determined?

50. If a cylinder is 18 in. in diameter, and the water at the side of the vessel a foot above the water at the axis, what number of turns a minute must the vessel make round its axis? *Ans.* 101·9.

51. In the last question, if the water were a foot deep when the vessel was at rest, what was its depth at the axis when the vessel was in motion?

Ans. 6 in.

52. If the vessel were exactly full and closed at top, show how to determine the pressure at any point of the top. Why should the result be independent of gravity?

53. A cylinder 2 ft. in radius is filled with water and closed; it makes 80 turns a minute round its axis, which is vertical; show that the pressure on the top at a point (1) distant 1 ft. from the axis at the rate of 15·2 abs. un. per square inch; (2) distant 2 ft. from the axis at the rate of 60·9 abs. un. per square inch.

54. How can the pressure at any point of the cylindrical surface be determined? In the last question, suppose the cylinder to be 2 ft. deep, find the pressure per square inch at a point (1) on the bottom below the axis; (2) on the bottom 12 in. from the axis; (3) on the side 1 ft. below the top ($g=32$). How would these results be affected if the rotation occurred at a place where gravity did not act?

Ans. (1) 27·8 abs. un.; (2) 43 abs. un.; (3) 74·8 abs. un.

55. State and prove Torricelli's Theorem. Define the theoretical outflow.

56. A basin has in it a hole an inch square; water in the basin is kept at a constant level of 9 ft. above the hole; what is the theoretical outflow in one hour? *Ans.* 600 cub. ft.

57. State what correction has to be applied to the theoretical ontflow in the case of a rectangular aperture of moderate size; and mention the cir-

cumstances presupposed in the correction. What would be the corrected outflow in the last question? *Ans.* About 360 cub. ft.

58. When a short pipe or ajutage is applied to the hole, give some particulars as to its effect on the outflow.

59. On considering the proof of Torricelli's Theorem, do you see any reasons why it should not give strictly correct results?

60. Investigate the time occupied in emptying a cylindrical vessel by means of a small hole in its bottom.

61. A cylinder, the area of whose cross section is 60 sq. ft., is filled with water to a depth of 12 ft. A hole is made in its bottom, whose effective area is 0·5 sq. in.; find after how long a time the depth of the water will be (1) 8 ft., (2) 4 ft. *Ans.* (1) 2746 sec.; (2) 6325 sec.

CHAPTER VIII.

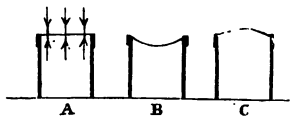
PNEUMATICS.

154. *The leading properties of air*, viz. that it is *heavy*, *fluid*, and *elastic*, are proved by many well-known experiments. We need not do more than mention briefly one experimental proof of each point. (a) As to the *weight* of air:—Take a glass globe furnished with a stop-cock, and weigh it; exhaust the air, and weigh it again. With a moderately sensitive balance the difference in weight will be very perceptible. Thus, suppose the globe to weigh 5600 gr., and to have an internal diameter of 3 in., the weight of the air withdrawn will, under ordinary circumstances, be about 4·2 gr., or about the 1-1400th part of the whole. The exact determination of the weight of a given volume of dry air is a very delicate operation. It has been found that 100 cub. in. of dry air weigh 30·940 gr., on the suppositions that the pressure equals that due to 30 in. of mercury in London, and that the temperature of both air and mercury is 60° F.[1] Without going into details, it is plain that as the atmosphere extends to a height of several miles, the pressure of the air on the surface of the earth must be very considerable, and, indeed, it is well known to be not much less than 15 lbs. per square inch. (b) As to the

[1] B. Stewart, *On Heat*, p. 70. The above determination is equivalent to the following: a litre of dry air at 0° C, and reduced to a pressure of 760 *millimètres of* mercury at Paris, is 1·29318 grammes.

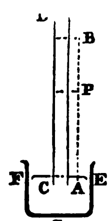
elasticity and *fluidity* of air:—Let A be a vessel of glass or earthenware whose mouth is covered air-tight with a piece of bladder. The atmosphere presses downward on the bladder with a considerable force, and this pressure is balanced by the reaction of the air within, as shown in the diagram A.

FIG. 141.



Withdraw some of the air, and the existence of the external pressure at once becomes apparent, for the bladder is pressed in, as shown at B; and that the pressure of the external air is that of a *fluid*, and is exerted equally in all directions, is shown by the fact that the bulge is unaffected when the vessel is held in a horizontal or in an inclined position. The existence of the reaction of the internal air is shown pretty plainly in B, but it is shown still more distinctly if the vessel is placed under the receiver of an air-pump, and the air partially withdrawn from around it. The result is shown at C; the external pressure being reduced, the bladder is bulged out by the pressure of the internal air. That these forces are very considerable is shown by exhausting still further the air from within B, and from around C—in either case the bladder is burst with explosive violence. The student must bear in mind that when air is enclosed in an air-tight vessel, it is as if a spring were shut up in it, and a spring which exerts its elastic force equally in all directions.

FIG. 142.



155. *The barometer.*—Let C D, a glass tube 34 in. or 35 in. long, closed at D, and open at C, be taken and filled with mercury; let the open end be stopped with a finger, and the tube placed in a vessel, E F G, containing mercury, as shown in the figure. On removing the finger some of the mercury will flow out of the tube into the

vessel, but a column of mercury about 30 in. high will remain in the tube, and above it a vacuum. The height is reckoned by the difference between the levels of the mercury in the tube and in the vessel, as indicated by the dotted line A B, and we will suppose it to be denoted in inches by h. Now, as the mercury in the tube communicates freely with the mercury in the vessel, the pressure at every point of the mercury along the line E F must be the same; but at some points this pressure is produced by the weight of the atmosphere, and at other points, viz. within the tube, it is produced by a column of mercury whose height is A B or h inches. We arrive, therefore, at the following conclusion:—Conceive a column of the atmosphere one square inch in cross section, the weight of the quantity of air in that column is equal to the weight of h cub. in. of mercury. We can further infer that if the height A B is observed to vary, there must be a corresponding variation in the weight of the superincumbent column of air. If we suppose any small quantity of air in the neighbourhood of the barometer to be enclosed without either rarefaction or condensation, it is said to exist under a pressure of h in. of mercury; and when h is about 30 in., it is said to be under a pressure of one atmosphere.

In order that the height A B (or h) may be an exact measure of the pressure of the air, several points have to be attended to:—(a) The space above the mercury, B D, must be entirely free from air or aqueous vapour. (b) The mercury must be pure. (c) It will be observed that when the height of the mercury in C D changes, the level of the mercury in E F will change too, though in a much less degree; consequently it is a point to be noticed that the measurement is to be made from the one surface to the other. (d) The height of the column A B is slightly less than it would be but for capillarity, and a small

addition has to be made to the measured height on this account. The quantity to be added is less as the diameter of the tube is larger, and is insensibly small when the diameter exceeds about $\frac{3}{4}$ of an inch. (*e*) The height must be reckoned with reference to a certain temperature of the mercury; the standard temperature being commonly taken to be 32° F. It is in fact plain that, since mercury expands and contracts as it becomes hotter and colder, the quantity of mercury above the level E F may be unchanged, and therefore the pressure on E F may be unchanged, and yet the height A B may vary solely in consequence of a change of temperature.

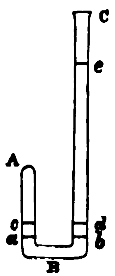
If, instead of mercury, water were used, we should have what is called a water barometer. As mercury has a specific gravity of about 13·6, the height of the column of water producing the same pressure as 30 in. of mercury is 408 in., or 34 ft. It may be remarked that the space above the mercury, though called the Torricellian vacuum, really contains vapour of mercury; but its elastic force is so small that it does not sensibly affect the height of the barometer. In the same manner the space above the water in a water barometer would contain vapour of water, which would have an elastic force depending on the temperature, and sufficient to lower the height of the column.

156. *Elasticity of air.*—Suppose a certain quantity of air to be enclosed in a vessel; let its volume be denoted by V, and let its elastic force, or the pressure which it exerts against each square unit of the surface of the containing vessel, be denoted by P. Now suppose that in any way, without changing its quantity, the volume is changed to v, and its pressure to p; then, assuming that the temperature is the same before and after the change of volume, the following relation will hold good:—

$$v : \text{V} :: \text{P} : p.$$

In other words, when a given quantity of air is at a constant temperature, its volume is inversely proportional to its pressure. This statement is sometimes called Boyle's and sometimes Mariotte's Law. Its truth may be verified by a series of experiments made as follows:—We will suppose that the barometer, at the time the experiments are made, stands at a height of h in. Let

FIG. 143.



A B C be a bent tube of uniform bore, the shorter leg of which, A B, is closed, and the longer, B C, open. Let mercury be poured in at C, and let it stand at equal heights in either leg, as at $a b$. A definite volume of air is thus enclosed in A a under a pressure equal to that of the external air, i.e. of h in. of mercury. Let more mercury be poured in at C; the level of the mercury will rise in both legs, but much more in B C than in B A. Suppose the surfaces to be at c and e. Through c draw the horizontal line $c\,d$. The air which formerly occupied the space A a now occupies the space A c; in the former case the pressure was equal to that due to h in. of mercury, it is now equal to that due to the weight of atmosphere and the column of mercury $e\,d$; or if $e\,d$ is k in. high, it is under a pressure equal to that of $h+k$ in. of mercury. The instrument is provided with a scale by which the lengths of A a, A c, $e\,d$, can be read off. It will be found that

$$A\,a : A\,c :: h+k : h.$$

Now, as the tube is of uniform bore, the spaces occupied by the air are in the same ratio as the volumes, and thus the verification of the law is complete.

It must be added that Boyle's Law is sensibly true for air and some other gases, unless the pressure to which it is subjected is very great, and even then the departure

from the law is but small. With gases which undergo liquefaction at moderate pressures, the departure from the law is greater, and increases as the state of liquefaction is approached. In what follows we shall assume that Boyle's Law is true without modification.

Ex. 168.—Suppose the barometer to stand at 30 in., and that A *a* was 8 in. long; suppose more mercury to be poured in till it reaches *c*, A *c* being 6 in., it would now be found that the mercury in B C is 10 in. above the level of the mercury in A *c*; consequently the pressure is now that due to 40 in. of mercury, and we see that

$$8 : 6 :: 40 : 30.$$

Ex. 169.—A piston 12 sq. in. in area works air-tight in a cylinder; there is a spring between the piston and the base of the cylinder, which is just in contact with the piston when it is 6 in. above the base. If half the air is withdrawn from the cylinder, and the piston falls only 2 in., what is the force with which the spring is now compressed? Barometer supposed to stand at 30 in.

The original quantity of air was 12×6 cub. in., and as half is withdrawn, there is now a quantity of air such as would occupy 12×3 cub. in., under a pressure of 30 in. of mercury; it however occupies 12×4 cub. in., and therefore its pressure is $\frac{3}{4}$ths of the atmospheric pressure; the remaining $\frac{1}{4}$th of the atmospheric pressure must be borne by the spring, which therefore is under a compressing force equal to that of $\frac{1}{4} \times 30 \times 12$ cub. in. of mercury, or about 44 lbs.

Ex. 170.—In fig. 142 suppose that A B is $29\frac{3}{4}$ in., and that B D is 5 in., the internal section of the tube being 1 sq. in.; a cubic inch of the external air is allowed to get into the tube; find the amount by which the surface of the mercury in the tube falls.

It is to be remarked that if any point P be taken in the column, the pressure at the point P is the pressure of the external air reduced by the pressure due to the column of mercury of the height A P. Bearing this in mind, suppose the surface of the mercury to fall x in., then the air that has been let in occupies $5 + x$ cub. in. of space, and is under a pressure of $29\frac{3}{4} - (29\frac{3}{4} - x)$, or x in. of mercury. The question, then, comes to this:—A cubic inch of air under a pressure of $29\frac{3}{4}$ in. of mercury occupies $5 + x$ cub. in.; under a pressure of x in. of mercury, what is x? Boyle's Law gives us

$$1 : 5 + x :: x : 29\tfrac{3}{4};$$

whence $x = 3\frac{1}{2}$ in., or the mercury falls $3\frac{1}{2}$ in.

157. *Relation between density and elastic force of air.*—If we take any quantity of air under a certain

pressure, and if we suppose its volume to be halved, its density is, of course, doubled; if we suppose its volume to be reduced to one-third of its original volume, its density is trebled. But we see from Boyle's Law that in the former case its pressure is doubled and in the latter trebled. Now, the same would be true in any proportion; consequently we can infer that the pressure of a gas is directly proportioned to its density, provided its temperature continues constant.

158. *The height of the homogeneous atmosphere.*—We have seen that the density of air is proportional to its elastic force, which is, of course, equal to the force that compresses it. We may, therefore, ask this question:—When air has a certain density (ρ), what must be the height of a column of air supposed to be of the same density (ρ) throughout, which would produce a pressure equal to the elastic force of the air? If we denote the height of this column by H, the pressure produced by its weight per unit of area will be $\text{H}\,\rho\, g$, and this equals the elastic force of air whose density is ρ. Suppose that, under these circumstances, the height of the barometer is h, then if the density of mercury is denoted by σ, the pressure due to the mercury per unit of area is $h\,\sigma\, g$; this also equals the elastic force of the air. We have, therefore,

$$\text{H}\,\rho\, g = h\,\sigma\, g, \text{ or } \text{H} = \frac{h\,\sigma}{\rho}.$$

If we had supposed the density of the air to be ρ_1, and the height of the barometer to be h_1, we should have found the height of the column of air to be $h_1\,\sigma \div \rho_1$. But we know from Boyle's Law that (Art. 157)

$$\frac{h\,\sigma}{\rho} = \frac{h_1\,\sigma}{\rho_1},$$

and consequently that the height denoted by H has the *same value* in the latter case as in the former. The

student must bear in mind that this reasoning goes upon the supposition that the temperatures of air and mercury and the force of gravity are the same in both cases. If we suppose the temperature both of air and mercury to be 32° F., and the force of gravity the same as at Paris (viz. $g = 32{\cdot}1819$), it can be shown that H equals 26,215 ft. This is called the height of the homogeneous atmosphere, and for this reason: whatever may be the density of the air, the pressure of the superincumbent air is the same as if the atmosphere extended upward with the same uniform density to a height of 26,215 ft., supposing temperature and force of gravity to be as stated.

The experimental determinations[1] from which the number 26,215 is inferred are as follows:—

1. One decimètre equals 3·93708 in., and a cubic decimètre of water at 4° C. weighs 1000 grammes.

2. Specific gravity, reckoned as in France, is the ratio of the mass of a given volume of solid or liquid at 0° C. to that of an equal volume of water at 4° C. The specific gravity of mercury is found to be 13·596. So that a cubic decimètre of mercury at 0° C. weighs 13,596 grammes.

3. At Paris, under a pressure of 7·6 decimètres of mercury at 0° C., one decimètre of dry air at 0° C. weighs 1·29318 grammes.

Hence H in decimètres equals $7{\cdot}6 \times 13{,}596 \div 1{\cdot}29318$, which is the same as 314,584 in. or 26,215 ft.

Ex. 171.—From the result given in the last article calculate the height of the homogeneous atmosphere at Greenwich, where g equals 32·1912.

If H is the height of the homogeneous atmosphere where the force of gravity is g, and if ρ is the density of the air, we have the compressing force due to the density equal to $H \rho g$. Similarly if H' is the height of the homogeneous atmosphere where the force of gravity is g', if ρ' is the density of the air, the compressing force is $H' g' \rho'$. Now if the temperatures are the same, the densities have the same ratio as the compressing force (Art. 157); or,

$$\rho : \rho' :: H g \rho : H' g' \rho'.$$

Therefore, $$H g = H' g'.$$

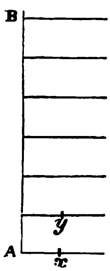
Hence $$H = 26{,}215 \times 32{\cdot}1819 \div 32{\cdot}1912 = 26{,}207.$$

159. *Determination of heights by the barometer.*—It will be supposed that the temperature of air and mercury

[1] Balfour Stewart, *On Heat*, pp. 68–71.

is 32° F. Let A and B be the stations, and let h_1 be the height of the barometer at A, and h_2 its height at B; the observations being, if possible, made simultaneously. Let the air be divided into a number of horizontal strata an inch thick, as shown in the figure; it is plain that, if we can find the number of these strata, we shall know the height of B above A in inches. Let $\rho, \rho_1, \rho_2, \rho_3 \ldots\ldots$ denote the density of the air in the successive strata beginning at A, and let H denote the height of the homogeneous atmosphere in inches. The pressure per square inch of the superincumbent atmosphere will be $g \mathrm{H} \rho$ at x, and $g \mathrm{H} \rho_1$ at y. Now the pressure at x must exceed that at y by the weight of a cubic inch of the air composing that stratum, i.e. by $g \rho$.

FIG. 144.



Hence, $$g \mathrm{H} \rho - g \rho = g \mathrm{H} \rho_1,$$

or $$(\mathrm{H}-1)\rho = \mathrm{H} \rho_1.$$

In just the same way it can be shown that

$$(\mathrm{H}-1)\rho_1 = \mathrm{H} \rho_2,$$

$$(\mathrm{H}-1)\rho_2 = \mathrm{H} \rho_3;$$

and if we multiply these equations together we obtain

$$(\mathrm{H}-1)^3 \rho = \mathrm{H}^3 \rho_3.$$

A similar result would be obtained if 4 or 5, or any other number of strata, had been considered; and hence if there are n strata between A and B, we have

$$(\mathrm{H}-1)^n \rho = \mathrm{H}^n \rho_n.$$

Now, as the densities are proportional to the pressures,

$$\rho : \rho_n :: h_1 : h_2;$$

and therefore, $$(\mathrm{H}-1)^n h_1 = \mathrm{H}^n h_2.$$

We know (Art. 158) that the height of the homogeneous atmosphere in inches is 314,584, and therefore the equation becomes

$$\left(\frac{314{,}584}{314{,}583}\right)^n = \frac{h_1}{h_2},$$

where n is the required height in inches. To find n take the logarithms on both sides, and we have

$$n\{\log.\ 314{,}584 - \log.\ 314{,}583\} = \log.\ h_1 - \log.\ h_2;$$

Now $\log.\ 314{,}584 - \log.\ 314{,}583 = 0{\cdot}0000013806$.

Hence $\qquad n = 724{,}350\{\log.\ h_1 - \log.\ h_2\}$;

or in feet, $\qquad \text{AB} = 60{,}360\ (\log.\ h_1 - \log.\ h_2)$.

Ex. 172.—On the suppositions of the last article let the height of the barometer at the lower station be 29·935 in., and at the upper station 27·875 in. What is the vertical height of the upper above the lower station?

We have

$$\begin{aligned} \log.\ 29{\cdot}935 &= 1{\cdot}4761793 \\ \log.\ 27{\cdot}875 &= 1{\cdot}4452149 \\ \hline &\ 0{\cdot}0309644 \end{aligned}$$

Hence the required height is $60{,}360 \times 0{\cdot}0309644$, or 1869 ft.

Remark 1.—To obtain a fairly exact value of the height it is necessary to apply a correction for the temperature of the air; this is done as follows:—Let $\text{T}_1{}^\circ$ and $\text{T}_2{}^\circ$ be the temperatures on Fahrenheit's scale at the upper and lower station; take $t = \frac{1}{2}(\text{T}_1 + \text{T}_2)$, and calculate the value of $1 + 0{\cdot}002036\ (t - 32)$. Multiply the height obtained in the last article by this factor and the result is the corrected height required. Thus, in the example of the last article, suppose the temperature of the air at the lower station to be 62° F., and at the upper station to be 48° F., the average temperature of the air is $\frac{1}{2}(62 + 48)$, or 55° F.; hence the required factor is $1 + 0{\cdot}002036 \times 23$, or 1·04683, and therefore the corrected height is $1869 \times 1{\cdot}04683$, or 1957 ft.

Remark 2.—The result obtained in the last article was arrived at by the use of a 10-figure logarithm; the use of such a logarithm can be easily avoided by a slight variation in the method; thus:—It is well known that

$$\log.\ (p+1) - \log.\ p = \log.\ \frac{p+1}{p} = \log.\ \left(1 + \frac{1}{p}\right) = \frac{1}{p} - \frac{1}{2}.\ \frac{1}{p^2} + \ \ldots\ .$$

Now if we suppose $\frac{1}{p}$ to be about 0·000003, we see that $\frac{1}{2}.\ \frac{1}{p^2}$ will be about

0·0000000000045, and consequently can be neglected if only the first ten places of decimals are required. Under these circumstances,

$$\log.(p+1)-\log.p=\frac{1}{p}.$$

This supposes that the logarithms are of the kind called hyperbolic or Napierian; but in the case of common logarithms we have

$$\log.(p+1)-\log.p=\frac{M}{p},$$

where M denotes the number 0·43429448. So far we have merely stated well-known facts about logarithms, and we see therefore that

$$\log.\ 314584-\log.\ 314583=\frac{M}{314583},$$

and consequently that the difference between the heights of the stations in inches is

$$\frac{314583}{0·42429448}(\log.h_1-\log.h_2),\text{ or }724350\,(\log.h_1-\log.h_2).$$

This way of looking at the matter is also useful for the following reason: we see that if H denotes, as before, the height of the homogeneous atmosphere, the required height of the upper station above the lower is

$$\frac{H}{M}(\log.h_1-\log.h_2),$$

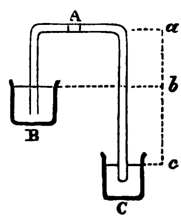
and therefore that every cause which produces a change in value of H will produce a proportionate change in the calculated differences of the heights of the stations. These causes are variations in the temperature of the air from 32° F., variations in the force of gravity from 32·1819, and the presence of moisture in the air. The most important is variation of temperature, and we have already seen how to correct for that; the others are of less importance and need not detain us.

160. There are several simple machines whose action depends on the properties of air and water which we have been engaged in studying; we will now describe some of them, quite briefly, but sufficiently to exhibit the principles involved in their construction and use. The student should refer to Art. 155, where he will find an explanation of what is meant by the 'water barometer.'

161. *The siphon.*—B and C are two vessels containing a fluid, say water, C being on the lower level; BAC, a bent tube with legs of unequal length filled with the fluid. If we suppose the ends of the tube BC to be open, the water will flow out of B through the tube into C. To explain this, draw

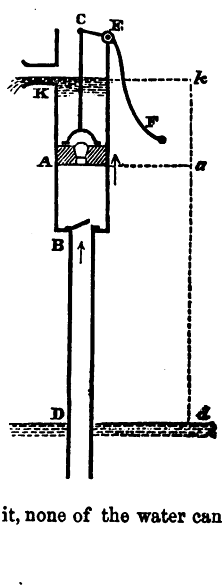
a vertical line $a\,b\,c$, and mark on it the points a, b, c, on the same level as A, and the surface of the water in B and C, also let h denote the height of the water barometer. Consider a small portion of the fluid at the point A; the pressure of the atmosphere on the surface of the water in B is transmitted to A, but is diminished by the weight of the column of water in the tube; consequently the portion of the fluid under consideration is urged in the direction A a by a force equal to the weight of a column of water having an equal cross section, and whose height is $h-a\,b$. For a like reason it will be acted on in the direction a A by a similar column whose height is $h-a\,c$. On the whole, therefore, it is acted on in the direction A a by a force equal to the weight of a similar column whose height is the excess of the former over the latter, i.e. $b\,c$, consequently it will begin to move in the direction A a. The atmospheric pressure on the surface of the water in B will prevent the formation of a vacuum at A and there will be a continuous flow of the water in the direction B A C. It will be observed (1) that the direction of the flow is wholly due to the fact that the level of the water in C is below that of the water in B; (2) that it is not necessary that the end of the tube at C should dip into the water; in that case the point c must be taken on a level with the end of the tube.

FIG. 145.



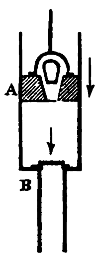
162. *The suction-pump.*—A is a piston attached by a rod A C to the end of a handle F E C, capable of turning on a fulcrum E, and thereby of working the piston up and down within the barrel. There is a valve in the piston which can open upwards, and at the top of the suction-tube B D there is a second valve which likewise opens upwards. To explain the action of the suction-pump, suppose the machine to be filled with water to the height K, on a level with the spout, and suppose the force applied at F and transmitted along the piston-rod to be in the act of raising the piston. The pressure on the top of A keeps its valve closed, and a vacuum would be formed below the piston were it not for the atmospheric pressure on the water in the well at D, which forces the water up the suction-tube, opens the valve at B, and keeps the barrel full. In consequence, the water above A will be raised and come out at the spout. Now suppose the action to be reversed. As soon as the piston begins to descend, the valve at B falls, and as it is being held down by the pressure above it, none of the water can

FIG. 146.



escape out of the barrel back into the suction-pipe; consequently the valve in the piston opens and the piston returns to the bottom of the barrel, while the water does not fall below the level K. The alternate upward and downward motion of the handle will thus cause a nearly continuous stream of water to come out of the spout.

FIG. 147.



To calculate the force which must act along the piston-rod to raise the piston we may proceed thus:—Draw the vertical line $k\,a\,d$ (fig. 146) and let the points k, a, d be on the same level as K, A, D respectively. The upper side of the piston sustains the weight of the water above it, and the pressure of the atmosphere. The pressure of the atmosphere on the surface of the well sustains the weight of the column A D, and consequently the pressure transmitted to the under side of the piston is the atmospheric pressure diminished by the weight of a column of water whose height is A D. Let h denote the height of the water barometer, A the area of the piston, and unity the weight of a cubic unit of water. Then, neglecting the thickness of the piston, the pressure downward on the top of the piston is

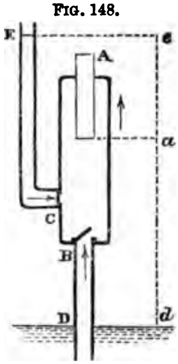
$$A\,h + A\,.\,k\,a,$$

and the pressure upward on the under side of the piston is

$$A\,h - A\,.\,a\,d.$$

The resistance to be overcome is the excess of the former force over the latter, i.e. $A\,.\,k\,d$, so that a force equal to this must act along the rod in order to raise the piston, e. g. if the area of the piston is 12 sq. in. and the height of the spout above the surface of the water in the well 20 ft., the required force is the weight of 12×240 cub. in. of water, say 104 lbs., supposing all frictions and resistances put out of the question.

FIG. 148.

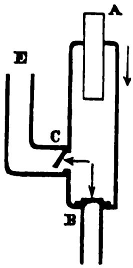


It will be observed that if the piston is at a height above D greater than the height of the water barometer, a vacuum will be formed below it. The pump therefore will not work when the piston is placed more than about 30 ft. above the surface of the water in the well.

163. *The forcing-pump.*—When water has to be raised to a height exceeding about 30 ft., the arrangement is commonly changed to that shown in the figure (148). A pipe C E is carried from the barrel to any required height; at C there is a valve which opens into the pipe; there is also a suction-pipe B D closed by a valve at B as before. The piston has no valve in it, and is often replaced by a solid cylinder A, called a plunger, which is worked by a handle as before. Suppose the whole to be full of water. If the plunger is raised, the pressure of the water in C E closes the valve at C; the

pressure of the atmosphere on the water in the well forces the water up the suction-pipe, opens the valve at B, and fills the barrel. When the plunger is forced down, the valve at B is closed, and the water, unable to escape into the suction-pipe, forces open the valve C, and escapes upward along C E (as shown in fig. 149). To ascertain the force required to work the pump, draw a vertical line and mark on it points e, a, d, on the levels respectively of the height to which the water is to be raised, the under side of piston or plunger, and the surface of water in well. Then, if A denotes the area of the cross-section of the plunger, A . ad and A . ae give the number of cubic units of water, whose weight equals the forces which will lift and force down the piston respectively. Thus let the cross section of the piston be 12 sq. in., and let it work at a height of 24 ft. above the surface of the water in the well, and 66 ft. below the point to which the water is to be raised; then to lift and to force down the plunger will require respectively forces equal to the weight of 12×288 and 12×792 cub. in. of water, i. e. about 125 lbs. and 343¾ lbs.

FIG. 149.



164. *The air-pump.*—A glass vessel or receiver A (fig. 150) fits air-tight on a smooth table; a pipe E F passes through the body of the table into a cylinder B, within which works a piston C; the end F of the pipe is closed by a valve at the end of the rod F G, which passes through the piston; the friction between the rod and the piston lifts the rod when the piston is raised, but a stop at G keeps the rod from rising through more than a very small space.

FIG. 150.

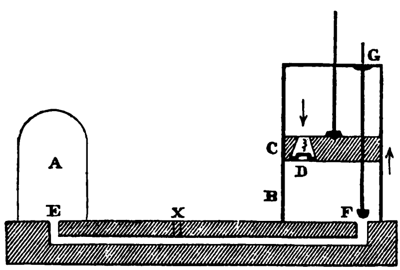
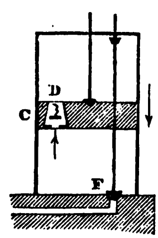


FIG. 151.



In the piston is a second valve D capable of opening upwards, but kept down by a delicate string. Suppose the piston to be in the act of being raised, the valve F is raised by the friction between the rod and the piston; a partial vacuum is formed below the piston, the air in the receiver filling the whole space, while the pressure of the external air keeps down the valve D. Now suppose the piston to be forced downward (fig. 151). The valve F is shut, so that the air in A continues in its rarefied state, while the

air in the cylinder below the piston is compressed until it opens the valve D, and escapes into the atmosphere; on the piston being raised a second time, the air in the receiver undergoes a further rarefaction; after a considerable number of up and down strokes a considerable degree of rarefaction is attained, and what is called a *vacuum* produced.

The air-pump is sometimes made in the form above described, but is then subject to the great inconvenience that in the upward motion the piston has a partial vacuum below it and above it the pressure of the atmosphere, so that great force is required to work the piston; there are ways in which this inconvenience can be avoided.

The degree of rarefaction produced by any given number of double, or up and down, strokes may be calculated thus:—Let A denote the volume of the receiver, including that of the pipe, and B that of the cylinder; let ρ, ρ_1, ρ_2, ρ_3 . . . . denote the densities of the air in the receiver, originally and after the first, second, third . . . . double stroke. Now as the quantity of air which originally had the volume A has the volume A + B, at the end of the first double stroke we must have (Art. 156)

$$\rho : \rho_1 :: A+B : A.$$

Similarly,
$$\rho_1 : \rho_2 :: A+B : A,$$

$$\rho_2 : \rho_3 :: A+B : A.$$

Therefore,
$$\rho : \rho_3 :: (A+B)^3 : A^3.$$

As it is plain that the like result would be true for any number of the double strokes, we must have

$$\rho : \rho_n :: (A+B)^n : A^n.$$

Thus, suppose that A is four times B, and we were required to find the density of the air in the receiver at the end of the fifteenth double stroke, we have

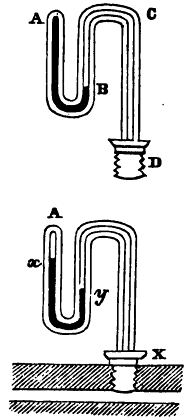
$$\rho_{15} = \rho \times (\tfrac{4}{5})^{15} = 0{\cdot}0352 \times \rho.$$

If the air originally had an elastic force equal to the pressure of 30 in. of mercury, this would give the elastic force of the air remaining in the receiver as equal to a pressure of 1·056 in. of mercury. Under the circumstances, it is frequently said that the *vacuum pressure* is one of 1·056 in. of mercury.

In order to measure the vacuum pressure a gauge is used, which in one of its forms may be described as follows:—A B C D is a glass tube with a double bend, closed at A and open at D; each leg is a few (say 6) inches long; at the open end D is a screw by which it may be fitted into a properly prepared opening (at X, fig. 150) in the pipe between the receiver and piston of the air-pump; it is thereby put into connection with the receiver. The leg A B is full of mercury, which is supported by the pressure of the

air on B. When the instrument is fitted into X, and the machine set in motion, no change is observed till the pressure is reduced to an amount less than that of the mercurial column A B, and then the mercury assumes the position seen at $x\,y$; the vacuum pressure is now measured by the height x above y.

FIG. 152.

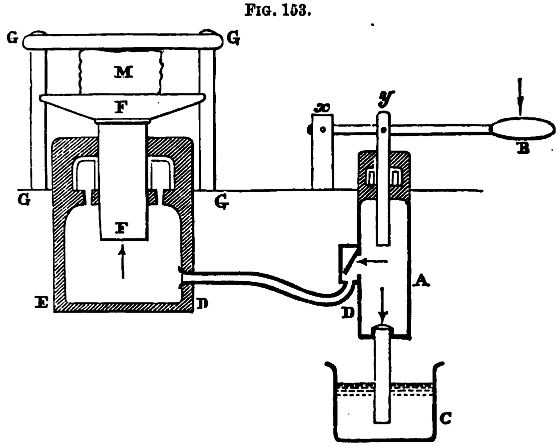


Besides giving an accurate measure of the degree of exhaustion actually obtained, the gauge supplies a means of ascertaining whether the machine is in good working order, and when the actual limit of the exhaustion has been reached.

It will be observed that if the machine acted with theoretical perfection, by taking n sufficiently large, ρ_n could be made less than any assignable magnitude. Practically, however, when a certain degree of exhaustion has been reached, no further effect is produced by working the instrument. It is said that with the best instruments the vacuum pressure can be reduced as low as 0·02 in. of mercury. There are other means by which the vacuum pressure can be reduced to as little as 0·00003 in. of mercury.

165. *The hydraulic press.*—A handle B works a force-pump A, by which water is drawn from a reservoir C, and forced through a pipe D D into a cylinder E; a piece F F works as a plunger in the cylinder and under the guidance of a framework G G G G. The machine may be employed in several ways; let us suppose it to be used for compressing a mass M placed between

FIG. 153.

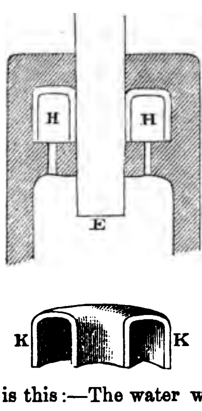


F and the framework. Suppose a force applied sufficient to make the handle move downward, the suction-pipe will be closed by its valve; the valve at D will be forced open, and water will be forced out of A into E, causing the plunger F to rise, and overcoming to a certain extent the resistance which the mass M offers to compression. If the pressure is taken off the handle, the valve at D falls, and the tendency of F to fall will cause a pressure against D, which will keep it closed, and thus none of the water can flow back out of E into A. When the handle is raised D is still closed, and water rises through the suction-pipe to fill the space that would be left void by the lifting of the plunger. This action may be continued until M has undergone a very great compression.

To calculate the pressure which may be brought to bear upon M, consider the following case:—Let $x\,y$ and x B equal 6 in. and 4 ft. respectively; the cross sections of the plungers in A and E be 1 sq. in. and 84 sq. in. respectively, and suppose the force at B to be 30 lbs. Then the force transmitted along the plunger from y is $30 \times 48 \div 6$, or 240 lbs. As this is exerted on an area of a square inch, the whole of the water in the cylinders and the pipe is in a state of stress at the rate of 240 lbs. per square inch, and consequently the pressure on the end of F is 240×84 lbs., or 9 tons; in other words, one man by pressing on the handle can easily bring to bear on M a crushing force of 9 tons, supposing the machine to act perfectly. It will be observed, however, that at each stroke F is raised through an exceedingly small space. Thus, if the end of the handle B works through 16 in., the point y will work through 2 in., and F will be raised 1–42nd of an inch.

It is plain that there is great liability to escape of water between the cylinders and the plungers, and particularly when considerable force is applied. The escape is prevented by the use of the cupped leather collar. A ring-shaped space is cut in the thickness of the cylinder surrounding the plunger, as shown at H H and communicating with the space within the cylinder by a few holes, so that the water would not only fill the space E, but also the space H H. A piece of leather is taken, saturated with grease, and then squeezed into this place so as to line three sides of it as indicated. The original form of this piece of leather is a circle with a circular hole cut in the middle, but after it is squeezed into its place, it takes the form shown by K K, which represents half of it. It will be observed that the collar is one continuous piece of leather without a seam. The end answered by the collar is this:—The water within the space H being in the same state of stress as the water in E, the greater the stress the more closely the leather

FIG. 154.



grasps the plunger, and the greater the resistance offered to the escape of the water.

It will be observed that when the plunger F is being forced up, all the water in A, E, and D D is in a state of stress, and each square inch of the interior surface of the cylinder and pipe is exposed to the same pressure, e. g. in the above example each square inch is under a pressure of 240 lbs., tending to burst these vessels. As the diameter of B is large, this cylinder must be made very strong to sustain the force tending to burst it; on the other hand, the diameter of D D may be very small, and consequently a comparatively weak pipe is strong enough to transmit the stress from A to B.

QUESTIONS.

1. What experimental proofs may be given of the fact that air is a heavy, elastic fluid?

2. Assuming that the pressure of the atmosphere is at the rate of 15 lbs. to the square inch, what is the pressure of the atmosphere on each face of a cubical box whose edges are a foot long? *Ans.* 2160 lbs.

3. If the lid of a box were 4 in. square, and fitted so tightly that no air could get out without lifting the lid, what weight must be put on it to keep it down when the box is put under the receiver of an air-pump, and the external air withdrawn? *Ans.* 240 lbs.

4. Describe the common barometer. What points have to be attended to in measuring the height of the column? What is meant by a water barometer, and what is its ordinary height? In what way do the vacuum spaces differ in the mercurial and water barometers?

5. State Boyle's Law, and describe its experimental verification.

6. State and prove the relation between the pressure and density of a gas—temperature being constant.

7. The mercurial column stands at 24 in., and has above it a vacuum space of 9 in., in which is some air; a barometer in perfect order reads 30 in. If the mercury in the latter instrument falls 1 in., what will the former instrument read? *Ans.* 23·384 in.

8. In what respects would the effects of a change of temperature on the two barometers in Q. 7 differ?

9. If a cubic foot of air were enclosed in a cubical vessel whose edge is 1 ft. long, when the barometer reads 30 in., and if it were possible for the force of gravity to change from 32 to 24, why should the pressure of the enclosed air on any face of the cube exceed that of the external air? What would be the excess? *Ans.* 16,829 abs. un.

10. There are two barometers on the sea-level, one in perfect order in which the mercury stands at a height of 30 in., the other with an imperfect vacuum of 9 in. above 25 in. of mercury. If it were possible for the force of gravity to change from 32 to 24, explain why the reading of the former barometer would undergo no change, while that of the latter would be changed from 25 in. to 24 in.

11. What is meant by the homogeneous atmosphere? When its height is said to be 26,215 ft. at Paris, what circumstances are supposed to exist? What are the experimental data from which this is inferred?

12. If H is the height of the homogeneous atmosphere when the force of gravity is g, and H′ its height when the force of gravity is g', show that $Hg = H'g'$.

13. If h_1 and h_2 are the heights of the barometer simultaneously at two stations A and B, give the reasoning by which it is shown that the vertical height of B above A is

$$60,360 \times \{\log. h_1 - \log. h_2\} \text{ feet.}$$

What suppositions as to the temperatures of air and mercury and the force of gravity are implied in this formula?

14. Explain how the correction for the temperature of air is applied to the barometric formula.

15. The height of the barometer at A is 29·835 in., and temperature of air 71° F.; at B the height of the barometer is 25·897 in., and temperature of air 43° F.; find the vertical height of B above A after correction for temperature of air. *Ans.* 3900 ft.

16. Change the form of the reasoning in Art. 159, so as to show that the number which multiplies ($\log. h_1 - \log. h_2$) is the height of the homogeneous atmosphere divided by a certain constant number.

17. Enumerate the circumstances which will affect the numerical determination of heights by the barometer.

18. Describe and explain the action of a common siphon. Why cannot water be raised from a lower to a higher level by a siphon? Why would the outflow be very slow if the levels of B and C (fig. 145) were nearly the same?

19. Describe and explain the action of the suction-pump. Calculate the force necessary to work the piston. If the diameter of the piston is 6 in. and the height of the spout above the well 25 ft., what force must act along the piston-rod to work the pump? *Ans.* 307 lbs.

20. Describe and explain the action of the forcing-pump. Explain the results stated in Art. 163 as to the force required to work the pump.

21. Describe and explain the action of the air-pump. Explain the *action of the* air-gauge and its use?

22. Calculate the vacuum pressure in the receiver of an air-pump, after a given number of strokes, supposing the instrument to be in perfect order.

23. Describe and explain the action of the hydraulic press. What means are used to prevent the escape of water at the spaces in which the plungers work?

24. Why may the pipe D D (fig. 153) be comparatively weak, though the water it contains transmits a force which causes an enormous pressure on M?

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